Northwestern University **Name:**

Math 290-1: Midterm 2

Fall Quarter 2014

Monday, November 17, 2014

Put a check mark next to your section:

Davis (10am)		Canez	
Alongi		Peterson	
Graham		Davis (12pm)	

Question	Possible points	Score
1	20	
2	20	
3	10	
4	15	
5	15	
6	20	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 10 pages, and 6 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **four** parts.)

   (a) Let A and B be two $n \times n$ matrices. If $\ker(A) = \ker(B)$, then A and B are both invertible.

Answer:

   (b) There exists a 6×7 matrix A with $\dim \ker A = 1$ and whose image is spanned by five linearly independent vectors.

Answer:

(c) If A is the 2×2 matrix of the reflection across a line through the origin in $\mathbb{R}^2$, then for any 2×2 matrix B we have

$$\det(AB^3AB) = \det(B^4).$$

Answer:

(d) Let $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis of $\mathbb{R}^3$ and suppose that $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^3$ for which $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Then $[\vec{x} + \vec{y}]_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Answer:

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **four** parts.)

   (a) Let P be a plane in $\mathbb{R}^3$. Then P is a subspace of $\mathbb{R}^3$.

Answer:

   (b) If A is an $n \times n$ matrix and if B is obtained from A by replacing the second row of A with

   $$(\text{first row of } A) - 2(\text{second row of } A),$$

   then $\det A = \det B$.

Answer:

(c) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linearly **dependent** set of vectors in $\mathbb{R}^m$, then $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is a linearly **independent** set of vectors in $\mathbb{R}^n$.

Answer:

(d) For a basis $\mathfrak{B}$ of $\mathbb{R}^n$, the expansion factor of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ in standard coordinates is equal to the expansion factor of the same linear transformation in $\mathfrak{B}$-coordinates (i.e. coordinates relative to $\mathfrak{B}$).

Answer:

3. Determine the values of a and b for which the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ 1 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \\ b \\ 0 \end{bmatrix}$$

are linearly independent.

4. Let V be the subspace of $\mathbb{R}^4$ consisting of all $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ satisfying both

$$x + 2y + 2z = 0 \text{ and } 3x + 6y + 7z - 3w = 0.$$

(a) Find the dimension of V.

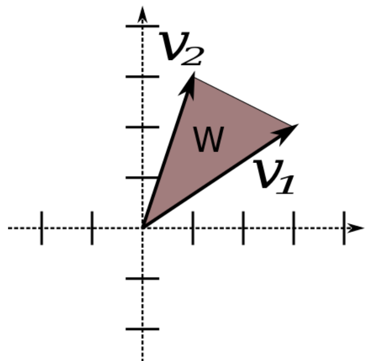
(b) Find a 4×4 matrix A whose image is V.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $y = 3x$.

(a) Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix B of T is diagonal, and compute B in this case.

(b) Using your answer to (a), compute the standard matrix (i.e. the matrix relative to the standard basis) A of T.

6. (This problem has **two** parts.) Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and let W be the shaded region in the diagram below.



(a) Calculate the area of W.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that is represented by the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$. Calculate the area of $T^3(W)$, the image of W under T^3, where T^3 denotes the composition of T with itself three times. Note that W denotes the same region as in part (a).