Northwestern University Name: SOLUTIONS
Student ID:

Math 290-1 Midterm 2

Autumn Quarter 2012

Monday, November 19, 2012

Put a check mark next to your section:

Allen		Cyr (12pm)	
Canez		Peters	
Cyr (10am)			

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 10 pages, and 7 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

Question	Possible points	Score
1	16	
2	16	
3	14	
4	14	
5	12	
6	14	
7	14	
TOTAL	100	

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

(a) The vector $\begin{pmatrix} 6 \\ 3 \\ 8 \\ 10 \end{pmatrix}$ is in the image of the matrix transformation determined by

$$\begin{pmatrix} 0 & 2 & 4 & 5 \\ 6 & 3 & 8 & 10 \\ 12 & 4 & 1 & 3 \end{pmatrix}$$

FALSE : image is a subspace of $\mathbb{R}^3$.
This vector is in $\mathbb{R}^4$.

(b) If A is an $n \times n$ matrix such that $\det(A^3) = 0$, then A is not invertible.

TRUE : $\det(A^3) = 0 \Rightarrow$
$\det(A)^3 = 0 \Rightarrow$
$\det(A) = 0$.
this is one of the equivalent conditions to non-invertibility.

(c) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that reflects a vector over the line $y = x$ and $\mathcal{B}$ is the basis $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$, then the matrix of T relative to $\mathcal{B}$ is $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

FALSE: $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$,

$T\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$

so the matrix of T rel. $\mathcal{B}$ is

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

(d) There is an invertible 2×2 matrix that sends the unit disc (in $\mathbb{R}^2$) to a subset of the y-axis.

FALSE: Say A is the matrix. Then

area(A(unit disk)) = |det A| · area (unit disk)

$\Rightarrow$

$$0 = |\det A| \cdot \pi$$

$\Rightarrow$ $\det A = 0$, so A is not invertible

or

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\y_1\end{pmatrix}$, $A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\y_2\end{pmatrix}$, so $A = \begin{bmatrix}0 & 0\\y_1 & y_2\end{bmatrix}$;

and rank $A < 2 \Rightarrow A$ not invertible.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

   (a) If $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\vec{w}$ is not a multiple of $\vec{v}_1$ or $\vec{v}_2$, then the set $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ is linearly independent.

SOMETIMES: $\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$.

$\vec{w} = \begin{pmatrix}1\\1\end{pmatrix}$ is not a multiple of $\vec{v}_1$ or $\vec{v}_2$,

but $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ is L.D.

On the other hand, $\vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ & $\vec{w} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ is an example where $\vec{w}$ is not a multiple of $\vec{v}_1$ or $\vec{v}_2$, and $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ is L.I.

   (b) Suppose A and B are 17×17 matrices and B is obtained from A by:

   first switching the 2^{nd} and the 17^{th} rows of A,
   then switching the 6^{th} and the 8^{th} rows of A,
   then switching the 10^{th} and the 11^{th} rows of A,
   then switching the 13^{th} and the 15^{th} rows of A,
   then switching the 9^{th} and the 14^{th} rows of A.

   Then

$$\det A = \det B.$$

SOMETIMES:

An odd number of swaps are performed,

so $\det A = -\det B$.

If $\det B \neq 0$, $\det B \neq -\det B$ so $\det A \neq \det B$.

If $\det B = 0$, $\det A = 0 = \det B$.

(c) Suppose A is a 2×2 matrix and let T be the linear transformation $T(\vec{x}) = A\vec{x}$. If $\mathcal{B}$ is a basis for $\mathbb{R}^2$ and $B = [T]_{\mathcal{B}}$ is the matrix of T relative to $\mathcal{B}$, then $\text{rank}(A) = \text{rank}(B)$.

ALWAYS:

Image (T) is the same no matter what co-ordinates you describe it with, so rank(A) = dim (Image (T)) = rank (B).

or

$B = [T]_{\mathcal{B}}$ means there is an invertible S with $B = S^{-1}AS$, so $\det(B) = \det(A)$. Either:

(1) $\det(A) \neq 0$, so $\det(B) \neq 0$ and $\text{rank}(A) = 2 = \text{rank}(B)$

(2) If $\det(A) = 0$, $\det(B) = 0$ too. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ iff $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(2a) If $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\text{rank}(A) = 0 = \text{rank}(B)$

(2b) If $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\text{rank}(A) = 1$ and $\text{rank}(B) = 1$

(d) A counterclockwise rotation (about the origin) of $\mathbb{R}^2$ has a standard matrix whose determinant is 1.

ALWAYS:

$$\det\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) = \cos^2\theta + \sin^2\theta = 1.$$

3. Find an expression for k in terms of a and b so that

$$\left\{\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}a\\b\\k\end{pmatrix}\right\}$$

is a linearly **dependent** set.

$$\begin{bmatrix}2 & 0 & a\\0 & 1 & b\\1 & 2 & k\end{bmatrix} \xrightarrow{\quad} \begin{bmatrix}2 & 0 & a\\0 & 1 & b\\0 & 2 & k-\frac{a}{2}\end{bmatrix} \xrightarrow{\quad} \begin{bmatrix}2 & 0 & a\\0 & 1 & b\\0 & 0 & k-\frac{a}{2}-2b\end{bmatrix}$$

$-\frac{1}{2}\times \mathrm{I}$ $\qquad$ $-2\times \mathrm{II}$

The set is L.D. if the last row is all zeros, ie $k-\frac{a}{2}-2b=0$, or

$$\boxed{k=\frac{a}{2}+2b}$$

4. Suppose A is an $n \times n$ matrix and $\vec{x}$ is some vector in $\mathbb{R}^n$ such that $A\vec{x} \neq \vec{x}$, but $A^2\vec{x} = A\vec{x}$. Explain how you know that $\ker A \neq \{\vec{0}\}$.

Since $A\vec{x} - \vec{x} \neq 0$,

but $A(A\vec{x} - \vec{x}) = A^2\vec{x} - A\vec{x} = 0$,

$A\vec{x} - \vec{x}$ is a non-zero vector in $\ker(A)$.

Thus $\ker(A) \neq \{\vec{0}\}$.

5. Let

$$A = \begin{pmatrix} -1 & 2 & 0 \\ -1 & -1 & 2 \\ 0 & 1 & -1 \end{pmatrix}.$$

Use the determinant to find a real number λ such that the matrix $(A - \lambda \cdot I_3)$ is not invertible.

$$\det(A - \lambda I_3) = \det\left(\begin{bmatrix} -1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 2 \\ 0 & 1 & -1-\lambda \end{bmatrix}\right) =$$

$$= \left(\left((-1-\lambda)^3 + 0 + 0\right) - \left(2(-1-\lambda) - 2(-1-\lambda) + 0\right)\right)$$

$$= (-1-\lambda)^3$$

If $(-1-\lambda)^3 = 0$, $\lambda = -1$.

So for $\boxed{\lambda = -1,}$ $A - \lambda I_3$ is not invertible.

6. Find the dimension of the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\underset{\vec{v}_1}{\begin{pmatrix}1\\-1\\4\\1\end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix}2\\-2\\8\\2\end{pmatrix}}, \underset{\vec{v}_3}{\begin{pmatrix}1\\0\\-2\\0\end{pmatrix}}, \underset{\vec{v}_4}{\begin{pmatrix}3\\-1\\-2\\1\end{pmatrix}}, \text{ and } \underset{\vec{v}_5}{\begin{pmatrix}4\\-2\\3\\0\end{pmatrix}}.$$

Note that $\vec{v}_2 = 2\times\vec{v}_1$, so we want to know the dimension of the subspace of $\mathbb{R}^4$ spanned by $\{\vec{v}_1, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$.

$$\begin{bmatrix}1 & 1 & 3 & 4\\ -1 & 0 & -1 & -2\\ 4 & -2 & -2 & 3\\ 1 & 0 & 1 & 0\end{bmatrix}\begin{matrix}\\ +I\\ -4\times I\\ -I\end{matrix} \longrightarrow \begin{bmatrix}1 & 1 & 3 & 4\\ 0 & 1 & 2 & 2\\ 0 & -6 & -14 & -9\\ 0 & -1 & -2 & -4\end{bmatrix}\begin{matrix}\\ \\ +6\times II\\ +II\end{matrix} \longrightarrow$$

$$\longrightarrow \begin{bmatrix}1 & 1 & 3 & 4\\ 0 & 1 & 2 & 2\\ 0 & 0 & -2 & 3\\ 0 & 0 & 0 & -2\end{bmatrix}$$

so $[\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4]$ has rank $=4$, thus the vectors are L.I. and their span is $\boxed{\text{4-dimensional}}$

7. Explain how you know that there can't be a 2×2 matrix A whose columns are orthogonal unit vectors, that sends the disc of radius 1 (centered at the origin in $\mathbb{R}^2$) to the disc of radius 2 (centered at the origin in $\mathbb{R}^2$).

The parallelogram determined by the columns of A is a square of side length 1, so $|\det(A)|$ = area (p.gram of columns) = $= 1$.

$|\det(A)|$ is an expansion factor, so the area of the image of the disc of radius 1 is equal to the area of the original, ie $\pi \cdot 1^2 = \pi$.

the area of the disk of radius 2 is $\pi \cdot 2^2 = 4$, so this cannot be the image under A of the disk of radius 1.