Solutions

Northwestern University

Name:

Student ID:

Math 290-1 Midterm Exam 2

Fall Quarter 2013

Monday, November 18, 2013

Put a check mark next to your section:

Allen		Cañez	
Broderick 10:00		Davis	
Broderick 12:00			

Question	Possible points	Score
1	18	
2	24	
3	16	
4	12	
5	15	
6	15	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 9 pages, and 6 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

   (a) If $(\vec{v}_1, \dots, \vec{v}_n)$ is a basis for $\mathbb{R}^n$ and A is an $n \times n$ matrix, then $(A\vec{v}_1, \dots, A\vec{v}_n)$ is a basis for $\mathbb{R}^n$.

False. Consider $A = [0]$. Then $A\vec{v}_1 = \dots = A\vec{v}_n = \vec{0}$ & $(0, 0, \dots, 0)$ is not a basis for $\mathbb{R}^n$ as it is linearly dependent.

   (b) If Ω is a region in $\mathbb{R}^2$ with nonzero area, and A is an invertible 2×2 matrix such that

$$\text{Area of } A(\Omega) = \text{Area of } \Omega,$$

   then $\det A = 1$.

False. Consider $\Omega = \{$unit square with corners $\vec{0}, \vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2\}$

& $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $A(\Omega) = \Omega$, so Area of $A(\Omega)$ = Area of Ω but $\det A = -1$.

(c) There is a vector $\vec{v} \in \mathbb{R}^3$ such that $\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \vec{v}\right)$ is a basis for $\mathbb{R}^3$.

True. Pick $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

We need to show $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are LI, as 3 LI vectors in $\mathbb{R}^3$ form a basis.

Linear independence $\Leftrightarrow$ $A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix}$ invertible $\Leftrightarrow$ $\det A \neq 0$.

So we can compute $\det A$ by expansion along 3rd column.

$\det A = \det \begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix} = -6$. So these vectors form a basis.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) For a reflection T across a line through the origin in $\mathbb{R}^2$, there is a basis of $\mathbb{R}^2$ relative to which the matrix of T contains a column of zeroes.

Never. Since T is invertible, the standard matrix for T has nonzero determinant, so every matrix similar to this one has nonzero determinant as well. In particular, if $\mathfrak{B}$ is a basis, then the matrix for T relative to $\mathfrak{B}$ cannot contain a column of zeroes.

(b) For a $\vec{b} \in \mathbb{R}^2$, the set of solutions $\vec{x} \in \mathbb{R}^3$ of the system $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix} \vec{x} = \vec{b}$ forms a subspace of $\mathbb{R}^3$.

Sometimes

If $\vec{b} = \vec{0}$, then the set of solutions is $\ker \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$, which is a subspace of $\mathbb{R}^3$.

If $\vec{b} \neq \vec{0}$, then since

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \vec{b},$$

$\vec{0}$ is not in the set of solutions, so this set cannot be a subspace.

(c) For a linear transformation $T : \mathbb{R}^{10} \to \mathbb{R}^7$, $\dim(\ker T) = 2$.

Never. By rank-nullity thm,

$\dim \operatorname{Ker} T + \dim \operatorname{im} T = 10.$

$\operatorname{im} T \subset \mathbb{R}^7 \Rightarrow \dim \operatorname{im} T \leq 7.$

So $\dim \operatorname{Ker} T \geq 3.$

(d) For fixed $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$, the kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ given by

$$T(\vec{x}) = \det \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{x} \\ | & | & | \end{bmatrix}$$

has dimension 2.

Sometimes. If $\vec{v}_1 = \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $T = 0$, and $\dim \operatorname{Ker} T = 3$.

If $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $T(\vec{x}) = x_3$, which has image $= \mathbb{R}$. By rank-nullity, $\dim \operatorname{Ker} T = 2$.

3. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix}$.

(a) Find a basis for $\ker A$.

We find $\ker A$ by solving $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & 1 & 4 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 4 & 2 & 0 \end{array}\right] \longrightarrow \begin{array}{c} -(\mathrm{II}) \\ \\ -2(\mathrm{II}) \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-t \\ -2s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$,

So $\left(\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$ is a basis for $\ker A$.

(b) Find a basis for $\operatorname{im} A$.

From part (a), $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which has pivots in the first and second columns.

Thus the first and second columns of A form a basis, so

$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right)$ is a basis for $\operatorname{im} A$.

4. Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that

$$\begin{bmatrix}1\\1\end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix}0\\3\end{bmatrix} \text{ and } \begin{bmatrix}-2\\3\end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix}-5\\1\end{bmatrix}.$$

Let $\mathcal{B} = (\vec{v_1}, \vec{v_2})$. We need

$$0\vec{v_1} + 3\vec{v_2} = \begin{bmatrix}1\\1\end{bmatrix},$$

so $$\vec{v_2} = \begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}.$$

We also need

$$-5\vec{v_1} + 1\cdot\vec{v_2} = \begin{bmatrix}-2\\3\end{bmatrix},$$

so $$-5\vec{v_1} = \begin{bmatrix}-2\\3\end{bmatrix} - \vec{v_2} = \begin{bmatrix}-2-\frac{1}{3}\\ 3-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}-\frac{7}{3}\\ \frac{8}{3}\end{bmatrix}$$

and $$\vec{v_1} = -\frac{1}{5}\begin{bmatrix}-\frac{7}{3}\\ \frac{8}{3}\end{bmatrix} = \begin{bmatrix}\frac{7}{15}\\ -\frac{8}{15}\end{bmatrix}.$$

Hence, $$\mathcal{B} = \left(\begin{bmatrix}\frac{7}{15}\\ -\frac{8}{15}\end{bmatrix}, \begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}\right).$$

5. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 3 & 0 & 1 & -2 \end{bmatrix}$$

(a) Compute $\det A$.

Proceed by cofactor expansion along the ~~third~~ second column.

$$\det A = -2 \cdot \det \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$

~~Ro~~ Now, expand along first row.

$$\det A = -2\left(\det\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} + 5 \det\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}\right) = -2\left(-5 + 5(-6)\right) =$$

$$= -2(5 \cdot (-1 + -6)) = 70$$

(b) How many solutions does the equation $A\vec{x} = \begin{bmatrix} -17 \\ 3 \\ 19 \\ -2 \end{bmatrix}$ have? Justify your answer.

$\det(A) \neq 0 \Rightarrow A$ invertible. So $A\vec{x} = \begin{bmatrix} -17 \\ 3 \\ 19 \\ -2 \end{bmatrix}$ has exactly one solution.

6. Find **ALL** values of c, k, and d, if any, for which span $\left(\begin{bmatrix}-1\\3\\2\end{bmatrix}, \begin{bmatrix}2\\-6\\c\end{bmatrix}, \begin{bmatrix}4\\k\\d\end{bmatrix}\right)$ has dimension

(*a*) 0 (*b*) 1 (*c*) 3.

Justify your answers.

(a): The subspace always includes $\begin{pmatrix}-1\\3\\2\end{pmatrix}$, so it never has dimension 0.

(b): We need both $\begin{pmatrix}2\\-6\\c\end{pmatrix}$ and $\begin{pmatrix}4\\k\\d\end{pmatrix}$ to be scalar multiples of $\begin{pmatrix}-1\\3\\2\end{pmatrix}$ This only happens when $c=-4$, $k=-12$, $d=-8$.

(c): We now want the vectors to be linearly independent. Equivalently, the matrix $\begin{bmatrix}-1 & 2 & 4\\3 & -6 & k\\2 & c & d\end{bmatrix}$ must be invertible. We start row-reducing:

$$\begin{bmatrix}-1 & 2 & 4\\3 & -6 & k\\2 & c & d\end{bmatrix}\begin{matrix}\\+3r_1\\+2r_1\end{matrix} \rightsquigarrow \begin{bmatrix}-1 & 2 & 4\\0 & 0 & k+12\\0 & c+4 & d+8\end{bmatrix} \rightsquigarrow \begin{bmatrix}-1 & 2 & 4\\0 & c+4 & d+8\\0 & 0 & k+12\end{bmatrix}$$

This is invertible when $c\neq-4$ and $k\neq-12$. (d can be any number.)