Northwestern University

Name:

Math 290-1: Midterm 2

Fall Quarter 2014

Monday, November 17, 2014

Put a check mark next to your section:

Davis (10am)		Canez	
Alongi		Peterson	
Graham		Davis (12pm)	

Question	Possible points	Score
1	20	
2	20	
3	10	
4	15	
5	15	
6	20	
TOTAL	100	

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 10 pages, and 6 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **four** parts.)

   (a) Let A and B be two $n \times n$ matrices. If $\ker(A) = \ker(B)$, then A and B are both invertible.

Answer: FALSE

   Take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

   Both $\ker A$ and $\ker B$ equal $\operatorname{Span}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ but neither are invertible.

   (b) There exists a 6×7 matrix A with $\dim \ker A = 1$ and whose image is spanned by five linearly independent vectors.

Answer: FALSE

   It must be true that

   $$\operatorname{rank} A + \dim \ker A = 7$$

   So $\operatorname{rank} A = \dim \operatorname{im} A = 6$.

   Thus need 6 linearly independent vectors to span the image of A.

(c) If A is the 2×2 matrix of the reflection across a line through the origin in $\mathbb{R}^2$, then for any 2×2 matrix B we have

$$\det(AB^3AB) = \det(B^4).$$

~~false~~

Answer:

$$\det(AB^3AB) = \det(A)\det(B^3)\det(A)\det(B)$$
$$= (\det(A))^2 \det(B^3)\det(B)$$
$$= (\det A)^2 \det(B^4)$$
$$= \det B^4$$

need to justify

The determinant of any reflection is -1, so $\det A = -1$.

A reflection across a line represented by A has the property that $A^2 = I$

$\det(I) = \det(A^2) = (\det A)^2 = 1$

(d) Let $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis of $\mathbb{R}^3$ and suppose that $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^3$ for which $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Then $[\vec{x} + \vec{y}]_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Using Definitions: Answer:

Since $[\vec{x}]_\beta = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ means that $\vec{x} = 1\vec{v}_1 + 1\vec{v}_2 + 1\vec{v}_3$

and $[\vec{y}]_\beta = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ means that $\vec{y} = -2\vec{v}_1 + 0\vec{v}_2 + 1\vec{v}_3$

So $\vec{x} + \vec{y} = (1-2)\vec{v}_1 + (1+0)\vec{v}_2 + (1+1)\vec{v}_3 = -\vec{v}_1 + \vec{v}_2 + 2\vec{v}_3$

mean $[\vec{x}+\vec{y}]_\beta = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

Easier Solution: $[\vec{x}+\vec{y}]_\beta = [\vec{x}]_\beta + [\vec{y}]_\beta$ since ~~the~~ coordinate transformations are linear.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **four** parts.)

   (a) Let P be a plane in $\mathbb{R}^3$. Then P is a subspace of $\mathbb{R}^3$.

   Answer: Sometimes

   If P contains the origin, then P is a subspace. (Ex: $x+y+z=0$)
   If P does <u>not</u> contain the origin, then
   P is <u>not</u> a subspace. (Ex: $x+y+z=1$)

   (b) If A is an $n \times n$ matrix and if B is obtained from A by replacing the second row of A with

   $$(\text{first row of } A) - 2(\text{second row of } A),$$

   then $\det A = \det B$.

   Answer: Sometimes

   Note that, if $A = \begin{pmatrix} \mathrm{I} \\ \mathrm{II} \\ \vdots \end{pmatrix}$, then $B = \begin{pmatrix} \mathrm{I} \\ \mathrm{I} - 2\,\mathrm{II} \\ \vdots \end{pmatrix}$,

   So $\det(B) = \det\begin{pmatrix} \mathrm{I} \\ -2\,\mathrm{II} \\ \vdots \end{pmatrix} = -2\det\begin{pmatrix} \mathrm{I} \\ \mathrm{II} \\ \vdots \end{pmatrix} = -2\det(A)$.

   So, $\det(A) = \det(B)$ if and only if $\det(A) = 0$.

   Example for True

   $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

   $\det(A) = 0 = \det(B)$

   Example for False

   $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$

   $\det(A) = 1 \neq -2 = \det(B)$

(c) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linearly **dependent** set of vectors in $\mathbb{R}^m$, then $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is a linearly **independent** set of vectors in $\mathbb{R}^n$.

Answer: Never

Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Assume that $\{\vec{v}_1, \ldots, \vec{v}_p\} \subseteq \mathbb{R}^m$ is linearly dependent. There are scalars $c_1, \ldots, c_p$, not all 0 such that $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$. Because T is linear,

$$T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) = T(\vec{0})$$

$$c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p) = \vec{0}$$

Because $c_1, \ldots, c_p$ are not all 0, $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is linearly dependent.

(d) For a basis $\mathfrak{B}$ of $\mathbb{R}^n$, the expansion factor of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ in standard coordinates is equal to the expansion factor of the same linear transformation in $\mathfrak{B}$-coordinates (i.e. coordinates relative to $\mathfrak{B}$).

Answer: Always

Method 1: the expansion factor is the ratio $\dfrac{\text{area}(T(\Omega))}{\text{area}(\Omega)}$.

This depends only on T, not on the coordinates chosen, so the expansion factor is the same in both cases.

Method 2: Let A be the matrix of T in standard coordinates,
and let B " " " " " in $\mathfrak{B}$-coordinates.
Since A and B are both matrices for the transformation T, there is an invertible matrix S such that $AS = SB$.
Thus A and B are similar, so their determinants are equal,

$$\det A = \det B.$$

Thus it is also true that $|\det A| = |\det B|$.
Since the expansion factor of the matrix A is $|\det A|$, and the " " " " B is $|\det B|$, the expansion factors are equal.

3. Determine the values of a and b for which the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ 1 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \\ b \\ 0 \end{bmatrix}$$

are linearly independent.

The vectors are linearly independent if and only if we can row reduce the following matrix to I_4:

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 3 & a & 2 \\ -2 & -1 & 1 & b \\ 4 & 2 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 3 & a-2 & 4 \\ 0 & -1 & 3 & b-2 \\ 0 & 2 & -6 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & a+7 & -2 \\ 0 & 0 & 0 & b \end{pmatrix}$$

We must have pivots in the last two rows, so we're looking for $a, b \in \mathbb{R}$: $\boxed{a \neq -7,\ b \neq 0}$

4. Let V be the subspace of $\mathbb{R}^4$ consisting of all $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ satisfying both

$$x + 2y + 2z = 0 \text{ and } 3x + 6y + 7z - 3w = 0.$$

(a) Find the dimension of V.

V is the null space of

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 6 & 7 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Two pivots, so the rank is 2, so the nullity is $4-2=2$ by Rank-Nullity Theorem.

$$\Longrightarrow \dim V = 2.$$

(b) Find a 4×4 matrix A whose image is V.

The rref above gives us the equations

$$x + 2y + 6w = 0$$
$$z - 3w = 0$$
w, y free

So V is all vectors of the form $y\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

A basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$. So we may take

$$A = \begin{bmatrix} -2 & -6 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(there are other answers).

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $y = 3x$.

(a) Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix B of T is diagonal, and compute B in this case.

Let L be the line $y = 3x$,
pick $\vec{v}_1 // L$ and $\vec{v}_2 \perp L$.
Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Pick $\vec{v}_2$ s.t. $\vec{v}_1 \cdot \vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.
so $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$

Since $T\vec{v}_1 = \vec{v}_1$, $T\vec{v}_2 = -\vec{v}_2$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

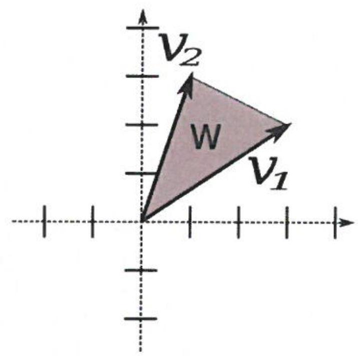
(b) Using your answer to (a), compute the standard matrix (i.e. the matrix relative to the standard basis) A of T.

$$S = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad S^{-1} = \frac{1}{\det S} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$
$$= \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}.$$

$$SA = SB$$

$$\Rightarrow A = SBS^{-1} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$
$$= \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

6. (This problem has **two** parts.) Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and let W be the shaded region in the diagram below.



(a) Calculate the area of W.

$$\text{area } W = \frac{1}{2}\left|\det \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}\right|$$

$$= \frac{1}{2}(9-2)$$

$$= \frac{7}{2}$$

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that is represented by the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$. Calculate the area of $T^3(W)$, the image of W under T^3, where T^3 denotes the composition of T with itself three times. Note that W denotes the same region as in part (a).

expansion factor of T = $|\det A| = |-3| = 3$

expansion factor of T^3 = $|\det A^3| = |\det A|^3 = 3^3 = 27$

area of $T^3(W)$ = ~~det~~ (expansion factor of T^3) area W

= $27\left(\frac{7}{2}\right)$