Northwestern University Name: KEY
Student ID:

Math 290-3 Midterm 2

Spring Quarter 2013

Thursday, May 23, 2013

Put a check mark next to your section:

Allen		Peters	
Canez			

Instructions:

- Read each problem carefully.
- Write legibly.
- Show all your work on these sheets.
- This exam has 9 pages, and 6 questions. Please make sure that all pages are included.
- You may not use books, notes or calculators.
- You have one hour to complete this exam.

Good luck!

Question	Possible points	Score
1	18	
2	18	
3	16	key
4	16	
5	16	key
6	16	
TOTAL	100	

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

   (a) $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = \int_0^{\pi}\int_0^{2\sin(\theta)} r\,dr\,d\theta$

TRUE both are equal to ~~B~~ π,
i.e. the area of a unit circle
(one centered at (0,0), second one
centered at (0,1))

   (b) Let $\mathbf{x} : [0,\pi] \to \mathbb{R}^2$ be the path $\mathbf{x}(t) = (2\cos(2t), 3\sin(2t))$, and let $\mathbf{y} : [0,\pi/4] \to \mathbb{R}^2$ be the path $\mathbf{y}(t) = (2\cos(2\pi - 4t), 3\sin(2\pi - 4t))$. Then, for any continuous vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\int_{\mathbf{y}} F \cdot d\mathbf{s} = -\int_{\mathbf{x}} F \cdot d\mathbf{s}.$$

FALSE they parametrize different paths:
$\vec{x}(0) = (2,0)$, $\vec{x}(\pi) = (2,0)$ but
$\vec{y}(0) = (2,0)$, $\vec{y}(\frac{\pi}{4}) = (-2,0)$.

Say eg $\vec{F} = (1,1)$ so $\vec{F} = \nabla(x+y)$.

Then $\int_{\vec{y}} \vec{F}\cdot d\vec{s} = f(2,0) - f(-2,0) = 4$

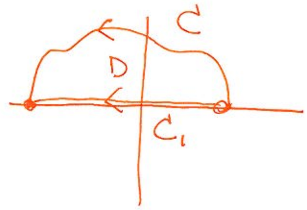
but $\int_{\vec{x}} \vec{F}\cdot d\vec{s} = f(2,0) - f(2,0) = 0$.

(c) For a simple curve C starting at $(1,0)$, ending at $(-1,0)$, and otherwise never crossing the x-axis, we have

$$\int_C [2xy\,dx + (x^2 + y\cos y + x)\,dy] = A + \int_{C_1} [2xy\,dx + (x^2 + y\cos y + x)\,dy],$$

where A is the area of the region between C and the x-axis, and C_1 is the line segment from $(1,0)$ to $(-1,0)$.

(ASSUMING C has $y \geq 0$)



TRUE: Green's Thm $\Rightarrow$

$$\int_{C + C_1^{opp}} (2xy\,dx + (x^2 + y\cos y + x)\,dy) = \iint_D (2x+1) - (2x)\,dA$$

$$\int_C (2xy\,dx + (x^2 + y\cos y + x)\,dy) - \int_{C_1} (2xy\,dx + (x^2 + y\cos y + x)\,dy) = \iint_D 1\,dA = A$$

& rearrange terms.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

   (a) Let A_a be the volume of the region between the two planes $z = a$ and $z = a + 3$, outside the cone $z^2 = x^2 + y^2$, and inside the hyperboloid $z^2 + 4 = x^2 + y^2$. For $a \geq 0$ and $b \geq 0$, we have $A_a = A_b$.

ALWAYS Using cylindrical coordinates, for any $a \geq 0$, we have

$$A_a = \int_0^{2\pi}\int_a^{a+3}\int_z^{\sqrt{z^2+4}} 1 r\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_a^{a+3} \frac{1}{2}r^2\Big]_{r=z}^{r=\sqrt{z^2+4}} dz\,d\theta$$

$$= \int_0^{2\pi}\int_a^{a+3} \frac{1}{2}(z^2+4-z^2)\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_a^{a+3} 2\,dz\,d\theta = \int_0^{2\pi} 2z\Big]_{z=a}^{z=a+3} d\theta = \int_0^{2\pi} 6\,d\theta = 12\pi.$$

This does not depend on a.

   (b) The path $\mathbf{x} : [a, b] \to \mathbb{R}^3$ given by $\mathbf{x}(t) = (3\cos(t), 5\sin(t), \cos(t))$ is a parametrization of a simple, closed, C^1 curve in $\mathbb{R}^3$.

SOMETIMES

$\vec{x}$ is C^1 for any $[a,b]$.

If $b = a + 2\pi$, then $\vec{x}$ is injective except that
$\vec{x}(b) = \vec{x}(a+2\pi) = \vec{x}(a)$

In this case $\vec{x}$ parametrizes a simple closed curve.

If $b < a + 2\pi$, then $\vec{x}(a) \neq \vec{x}(b)$, so $\vec{x}$ parametrizes a nonclosed curve.

It is also false if $b > a + 2\pi$, since then it will be non-injective at points other that a and b.

(c) Suppose that the vector field $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is defined and continuous on an open region in $\mathbb{R}^2$ containing the simple curve C. If curl$\mathbf{F} = \mathbf{0}$, then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

SOMETIMES

- True if C is closed and the region F is defined on is simply connected.
- False if C is not closed, eg

$$F(x,y) = (x, y) \quad \left(\text{note } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0\right)$$

and $C = (t, 0) \quad 0 \le t \le 1$:

$$\int_C \vec{F} \cdot d\vec{s} = \int_0^1 (t,0) \cdot (1,0)\, dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2} \neq 0$$

KEY

3. Let D be the quadrilateral region whose vertices are $(0,0)$, $(1,2)$, $(4,3)$, and $(3,1)$. Use the substitution $u = 2x - y$ and $v = 3y - x$ to rewrite

$$\iint_D \sin(2x - y)\cos(3y - x)\,dxdy$$

as an integral in u and v. You do not have to evaluate the resulting integral.

$$\left.\begin{aligned} u &= 2x - y \\ v &= 3y - x \end{aligned}\right\} \rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}}_{J^{-1}} \begin{pmatrix} x \\ y \end{pmatrix}$$

J^{-1}, the inverse of the Jacobian.

$\det(J) = (\det(J^{-1}))^{-1} = 5^{-1}$

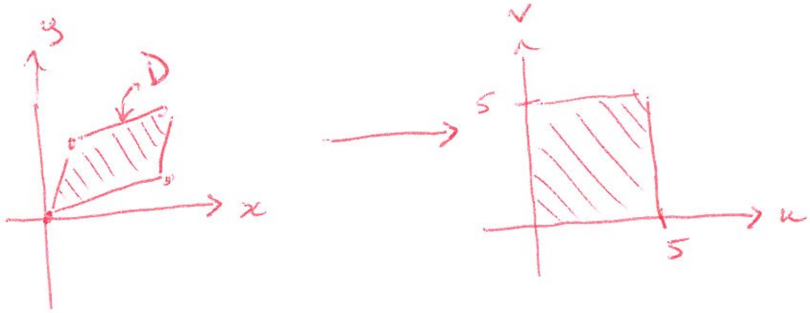
So,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \xmapsto{J^{-1}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \xmapsto{J^{-1}} \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \xmapsto{J^{-1}} \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \xmapsto{J^{-1}} \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$



$$\iint_D \sin(2x-y)\cos(3y-x)\,dxdy = \int_0^5 \int_0^5 \det(J) \sin u \cos v \,du\,dv$$

$$\boxed{= \int_0^5 \int_0^5 \frac{1}{5} \sin u \cos v \,du\,dv}$$

4. Let C be the intersection of the cylinder $\frac{x^2}{25} + \frac{z^2}{9} = 1$ and the plane $3y + 4z = 0$. Find

$$\int_C \left(\frac{x^2}{5} - yz - z^2 \right) ds.$$

parametrize $\vec{x}(t) = (5\cos t, -\frac{4}{3}(3\sin t), 3\sin t)$, $0 \le t \le 2\pi$

$= (5\cos t, -4\sin t, 3\sin t)$.

Then $\displaystyle\int_C \frac{x^2}{5} - yz - z^2 \, ds =$

$$= \int_0^{2\pi} \frac{25\cos^2 t}{5} + 12\sin^2 t - 9\sin^2 t \cdot \sqrt{25\cos^2 t + 16\sin^2 t + 9\sin^2 t} \, dt$$

$$= \int_0^{2\pi} 5\left(\frac{5}{2}(1 + \cos 2t) + \frac{3}{2}(1 + \sin 2t)\right) dt$$

$$= 5\left(\frac{5}{2}t + \frac{5}{4}\sin 2t + \frac{3}{2}t - \frac{3}{4}\cos 2t\right)\Big|_0^{2\pi}$$

$$= 5(5\pi + 3\pi) = 40\pi$$

5. Let $\mathbf{F}(x, y) = y^2z^3\mathbf{i} + 2yz(xz^2 + e^{y^2z})\mathbf{j} + (y^2e^{y^2z} + 3xy^2z^2)\mathbf{k}$.

(a) Determine whether or not $\mathbf{F}$ is conservative. Justify your answer.

F is continuous and differentiable on $\mathbb{R}^3$, so can check $\nabla \times F = 0$:

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y^2z^3 & 2xyz^3 + 2yze^{y^2z} & y^2e^{y^2z} + 3xy^2z^2 \end{vmatrix}$$

$$= i\left(2ye^{y^2z} + y^2z(2y)e^{y^2z} + 6xyz^2 - 6xyz^2 - 2ye^{y^2z} - 2y^3ze^{y^2z}\right)$$

$$- j\left(3y^2z^2 - 3y^2z^2\right) + k\left(2yz^3 - 2yz^3\right) = 0.$$

Therefore, conservative.

(b) Let $\mathbf{x}(t) = (\sin t, t, \cos t)$, with $-2\pi \leq t \leq 6\pi$. Compute the line integral of $\mathbf{F}$ along the path $\mathbf{x}$.

F conservative $\Rightarrow F = \nabla f$ for some $f : \mathbb{R}^3 \to \mathbb{R}$.

$\Rightarrow \partial_x f = y^2z^3, \quad \partial_y f = 2xyz^3 + 2yze^{y^2z}, \quad \partial_z f = y^2e^{y^2z} + 3xy^2z^2$

$f = xy^2z^3 + C_1(x,y) \qquad f = xy^2z^3 + e^{y^2z} + C_2(x,z) \qquad f = xy^2z^3 + e^{y^2z} + C_3(x,y$

so $f = xy^2z^3 + e^{y^2z}$ satisfies $F = \nabla f$.

Then, $$\int_{x(t)} F \cdot ds = \int \nabla f \cdot ds = f(x(6\pi)) - f(x(-2\pi))$$

$$= f(0, 6\pi, 1) - f(0, -2\pi, 1) = \boxed{e^{36\pi^2} - e^{4\pi^2}}$$

6. Suppose that $\mathbf{F}(x, y) = \left(-\frac{y^3}{12} + x^{e^{\cos x}}\right)\mathbf{i} + \left(e^{e^y} + \frac{x^3}{27}\right)\mathbf{j}$. Compute the line integral of $\mathbf{F}$ over the ellipse $4x^2 + 9y^2 = 36$ oriented clockwise.

Use Green's Theorem (with a negative due to orientation):

$$\oint_C \vec{F} \cdot d\vec{s} = -\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

where $M = \frac{-y^3}{12} + x^{e^{\cos x}}$, $N = e^{e^y} + \frac{x^3}{27}$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{x^2}{9} - \left(-\frac{y^2}{4}\right) = \frac{x^2}{9} + \frac{y^2}{4}$$

$$\oint_C \vec{F} \cdot d\vec{s} = -\iint_D \left(\frac{x^2}{9} + \frac{y^2}{4}\right) dA.$$

Use change of variables: $x = 3r\cos\theta$
$y = 2r\sin\theta$

Jacobian is $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} 3\cos\theta & -3r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{vmatrix} = 6r$

So $\oint_C \vec{F} \cdot d\vec{s} = -\iint_D \left(\frac{x^2}{9} + \frac{y^2}{4}\right) dA$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$ so

$$= -\int_0^{2\pi}\int_0^1 r^2 \, 6r \, dr\, d\theta$$

$$= -6\int_0^{2\pi}\int_0^1 r^3 \, dr\, d\theta = \boxed{-3\pi}$$