Northwestern University

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Math 290-3 Midterm Exam 2 Solutions

Spring Quarter 2014

Thursday, May 22, 2014

1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

(a) If the force field $\mathbf{F}(x,y,z) = (xy, -3yz, y^2)$ acts on an object, moving it from $(0,0,0)$ to $(1,1,-1)$ along the path $(t^2, t, -t^3)$, then the work done by the force field is

$$\int_0^1 (t^3, 3t^4, t^2) \cdot (t^2, t, -t^3)\, dt.$$

FALSE: The correct vector line integral is

$$\int_0^1 \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt = \int_0^1 (t^3, 3t^4, t^2) \cdot (2t, 1, -3t^2)\, dt = \int_0^1 2t^4\, dt = \frac{2}{5}.$$

The given integral is $\int_0^1 3t^5\, dt = \frac{1}{2}$.

(b) If C is the oriented path parametrized by $\mathbf{x}(t) = (4\cos t, 4\sin t)$, for $\pi \le t \le 3\pi$, and $\mathbf{F}(x,y) = (e^{x^2} + 2y, 3x - \sin(\cos y))$, then

$$\oint_C F \cdot d\mathbf{s} = 16\pi.$$

TRUE: The path $\mathbf{x}$ parametrizes the circle of radius 4 centered at the origin, oriented counterclockwise, so if we let D be the disk enclosed by this curve, then by Green's theorem,

$$\begin{aligned}\oint_C F \cdot d\mathbf{s} &= \iint_D \left(\frac{\partial}{\partial x}(3x - \sin(\cos y)) - \frac{\partial}{\partial y}(e^{x^2} + 2y) \right) dA = \iint_D (3-2)\, dA \\ &= \text{Area } D = \pi(4)^2 = 16\pi.\end{aligned}$$

(c) If $\mathbf{F}$ is a continuous vector field on $\mathbb{R}^2$ such that $\mathbf{F}(x, y)$ is orthogonal to the vector $x\mathbf{i} + y\mathbf{j}$ for each (x, y) and C is parametrized by $\mathbf{x}(t) = (t, t)$, for $1 \le t \le 4$, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = 0.$$

TRUE: By assumption $\mathbf{F}(t, t)$ is orthogonal to (t, t) and therefore also orthogonal to $(1, 1) = \mathbf{x}'(t)$ for each $1 \le t \le 4$, so

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_1^4 \mathbf{F}(t, t) \cdot (1, 1)\, dt = \int_1^4 0\, dt = 0.$$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) Given an integer $n > 0$, the vector field

$$\mathbf{F}(x, y, z) = (9e^{\cos(nx+z)}, y^n z^{n+1}, y^{n+1}z^n + ne^{\cos(nx+z)})$$

has path-independent line integrals.

SOMETIMES: We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 9e^{\cos(nx+z)} & y^n z^{n+1} & y^{n+1}z^n + ne^{\cos(nx+z)} \end{vmatrix} = ((n+1)y^n z^n - (n+1)y^n z^n)\mathbf{i}$$
$$- (n^2(-\sin(nx+z))e^{\cos(nx+z)} - 9(-\sin(nx+z))e^{\cos(nx+z)})\mathbf{j} + (0-0)\mathbf{k}$$

The first and third components are zero for any n, but for the second to be constantly zero we need $n^2 - 9 = 0$. So it's true when $n = 3$, false otherwise.

(b) Given a continuous vector field $\mathbf{F}$ on $\mathbb{R}^2$ and a C^1 oriented curve C such that $\int_C \mathbf{F} \cdot d\mathbf{s} > 0$, $\mathbf{F}$ has the property that at every point along C, $\mathbf{F}$ points in the same direction as the tangent vector to C at that point.

SOMETIMES: Let $\mathbf{F}(x, y) = (x, 0)$ and let $\mathbf{x}(t) = (t, 0)$. Then if $1 \le t \le 3$,

$$\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = (t, 0) \cdot (1, 0) = t > 0$$

for all $1 \le t \le 3$ and we have $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_1^3 t\,dt > 0$. So this is an example where the statement is true. If we use the exact same vector field and the same $\mathbf{x}(t)$ but let t range over $[-1, 3]$, then at, for example, $t = -1$, $\mathbf{F}(\mathbf{x}(t)) = (-1, 0)$ which forms an angle of $180°$ with $(1, 0) = \mathbf{x}'(t)$, and yet

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{-1}^3 t\,dt = 9/2 - 1/2 = 4 > 0.$$

So this provides an example where the statement is false.

(c) Given two simple and closed C^1 paths C_1 and C_2 in $\mathbb{R}^2$ that do not pass through the origin and are both oriented clockwise, we have

$$2\oint_{C_2} \mathbf{F}\cdot d\mathbf{s} < \oint_{C_1} \mathbf{F}\cdot d\mathbf{s} < 3\oint_{C_2} \mathbf{F}\cdot d\mathbf{s}$$

where $\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

NEVER: There are two possibilities for each curve C as above: It either encloses the origin or does not. If it does not, then $\mathbf{F}$ is C^1 on the region enclosed by C, so by Green's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \pm\iint_D \operatorname{curl}\mathbf{F}\,dA,$$

where D is the region enclosed by C. But

$$\operatorname{curl}\mathbf{F} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = 0,$$

so $\oint_C \mathbf{F}\cdot d\mathbf{s} = 0$. If C does enclose the origin, then applying Green's Theorem implies that $\oint_C \mathbf{F}\cdot d\mathbf{s} = \oint_{C'} \mathbf{F}\cdot d\mathbf{s}$, where C' is the clockwise path around the unit circle, which is parametrized by $(\cos(-t), \sin(-t))$, $0 \le t \le 2\pi$. But

$$\oint_{C'} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} (-\sin(-t), \cos(-t))\cdot(\sin(-t), -\cos(-t))\,dt = \int_0^{2\pi} -1\,dt = -2\pi.$$

Thus, each of the above integrals is either -2π or 0. Since $0 \le 2(0)$, $-2\pi \ge 3(-2\pi)$, $0 \ge 3(-2\pi)$, and $-2\pi \le 2(0)$, the inequality is never satisfied.

3. Compute the scalar line integral of $f(x, y) = xy^2 - 2x$ over the piece of the circle $x^2 + y^2 = 9$ that lies in the first quadrant.

ANSWER: We can parametrize this curve with $\mathbf{x}(t) = (3\cos t, 3\sin t)$ for $0 \le t \le \pi/2$. By definition the scalar line integral is then

$$\begin{aligned}\int_0^{\pi/2} f(3\cos t, 3\sin t)\|\mathbf{x}'(t)\|\,dt &= \int_0^{\pi/2} [27(\cos t)(\sin^2 t) - 6\cos t]\sqrt{9\cos^2 t + 9\sin^2 t}\,dt \\ &= 3\int_0^{\pi/2} 27(\cos t)(\sin^2 t) - 6\cos t\,dt = 3\left[9\sin^3 t - 6\sin t\right]_0^{\pi/2} \\ &= 3(9-6) - 0 = 9.\end{aligned}$$

4. Xerxon the alien is in a spaceship when it suddenly begins to malfunction. At that moment, the ship is at position $(0, 0, 0)$, moving at a speed of $5\,km/s$ in the direction of the vector $(0, 0, -1)$. (Here, distances are in kilometers and time is in seconds.) Its malfunctioning boosters then provide the erratic acceleration

$$\mathbf{a}(t) = (\pi \sin(\pi t), \pi^2 \cos(\pi t), 2) \text{ (in } km/s^2).$$

(a) There is a small piece of space junk at the position $(3, 2, -6)$. When will Xerxon's spaceship collide with it?

ANSWER: We antidifferentiate to find Xerxon's position function. First, we have $\mathbf{v}(t) = (-\cos(\pi t) + c_1, \pi(\sin(\pi t) + c_2, 2t + c_3)$. The spaceship's initial velocity is $(0, 0, -5) = \mathbf{v}(t) = (-1 + c_1, 0 + c_2, 0 + c_3)$, so we have

$$\mathbf{v}(t) = (-\cos(\pi t) + 1, \pi(\sin(\pi t), 2t - 5).$$

Antidifferentiating again we get $\mathbf{x}(t) = (\frac{-1}{\pi}\sin(\pi t) + t + k_1, -\cos(\pi t) + k_2, t^2 - 5t + k_3)$. And the initial position is $(0, 0, 0) = \mathbf{x}(0) = (k_1, -1 + k_2, k_3)$, so

$$\mathbf{x}(t) = \left(\frac{-1}{\pi}\sin(\pi t) + t, -\cos(\pi t) + 1, t^2 - 5t\right).$$

To find when this is equal to $(3, 2, -6)$ we solve $t^2 - 5t = -6$, or $t^2 - 5t + 6 = 0$, which factors to $(t-2)(t-3) = 0$. So there are only two possibilities. If $t = 2$, then $x(2) = \frac{-1}{\pi}\sin(2\pi) + 2 = 2 \neq 3$, so the first coordinate is not correct. If $t = 3$, then $x(3) = \frac{-1}{\pi}\sin(3\pi) + 3 = 3$ and $y(3) = -\cos(3\pi) + 1 = 1 + 1 = 2$, so $\mathbf{x}(t) = (3, 2, -6)$. He hits the space junk after 3 seconds.

(b) If the spaceship hits the junk while moving faster than $2\,km/s$, it will damage the ship. Will Xerxon's spaceship be damaged?

ANSWER: His speed at the time of impact is

$$||\mathbf{v}(3)|| = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5} > \sqrt{4} = 2,$$

so Xerxon's ship will be damaged.

5. Consider the conservative vector field

$$\mathbf{F} = (yz^2 + e^{x-y}\cos z)\mathbf{i} + (xz^2 - e^{x-y}\cos z)\mathbf{j} + (2xyz - e^{x-y}\sin z + 1)\mathbf{k}$$

(a) Find a potential function for **F**.

ANSWER: We must have $f_x = yz^2 + e^{x-y}\cos z$, so $f = \int f_x\,dx = xyz^2 + e^{x-y}\cos z + g(y,z)$. Then

$$xz^2 - e^{x-y}\cos z = f_y = xz^2 - e^{x-y}\cos z + g_y,$$

so $g(y,z) = h(z)$. Also,

$$2xyz - e^{x-y}\sin z + 1 = f_z = 2xyz - e^{x-y}\sin z + h'(z),$$

so $h(z) = z + c$, so $f(x,y,z) = xyz^2 + e^{x-y}\cos z + z$ is a potential function.

(b) Compute the vector line integral of **F** over the curve consisting of the helix with parametric equations $\mathbf{x}(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$ followed by the line segment from $(1, 0, 2\pi)$ to $(0, 0, 0)$.

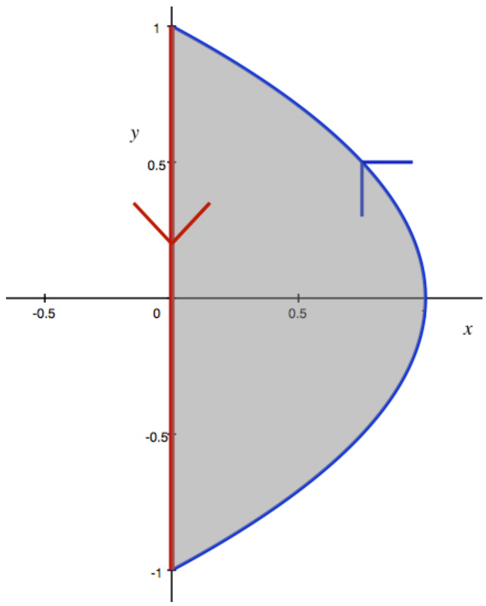
ANSWER: The path begins at the point $(1, 0, 0)$ and ends at the point $(0, 0, 0)$, so since $\mathbf{F} = \nabla f$, the integral is

$$f(0,0,0) - f(1,0,0) = (0 + 1 + 0) - (0 + e + 0) = 1 - e.$$

6. Let $\mathbf{F}(t) = (e^{e^x} + xy^2 + y, e^y + x^2y)$ and let C be the part of the parabola $x = 1 - y^2$ to the right of the y-axis, oriented counterclockwise. Find $\int_C \mathbf{F} \cdot d\mathbf{s}$.

ANSWER: The curl of $\mathbf{F}$ is much simpler than $\mathbf{F}$ itself, so we would like to apply Green's Theorem. The given curve is not closed so we have to "close it off" with the line segment $\mathbf{x}(t) = (0, -t)$ for $-1 \le t \le 1$.

In the picture on the right, the blue curve is C, the red curve is the path $\mathbf{x}$ we've added to "close it off," and the shaded region is the region enclosed by this new simple, closed, piecewise C^1 curve. Since it is oriented so that D lies to our left as we traverse the curve, Green's Theorem implies



$$\int_C \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA$$

So to compute the given integral, we need only compute the vector line integral along $\mathbf{x}$ and the integral of the curl over D.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{-1}^{1} (e^{e^0} + 0 - t, e^{-t} + 0) \cdot (0, -1) \, dt = \int_{-1}^{1} -e^{-t} \, dt = [e^{-t}]_{-1}^{1} = 1/e - e.$$

$$\begin{aligned}
\iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA &= \iint_D \left(\frac{\partial}{\partial x}(e^y + x^2y) - \frac{\partial}{\partial y}(e^{e^x} + xy^2 + y) \right) dA = \iint_D (2xy - 2xy - 1) \, dA \\
&= \int_{-1}^{1} \int_0^{1-y^2} -1 \, dx \, dy = \int_{-1}^{1} (y^2 - 1) \, dy = \left[\frac{y^3}{3} - y \right]_{y=-1}^{1} \\
&= (1/3 - 1) - (-1/3 + 1) = 2/3 - 2 = -4/3.
\end{aligned}$$

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA - \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = -4/3 - (1/e - e) = e - 1/e - 4/3.$$