1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

   (a) The distance (in the plane) between the point $(4,2)$ and the line $y = 3x + 5$ is the same as the distance (in $\mathbb{R}^3$) between the point $(4,2,1)$ and the plane $y = 3x + 5$.

TRUE. Let $\vec{n}_1 = (3,-1)$. Then $\vec{n}_1$ is orthogonal to the line $y=3x+5$. Since $(0,5)$ is on the line, we have

Distance from $(4,2)$ to the line $y = 3x+5$

$$= \left\| \frac{\vec{n}_1 \cdot (4-0, 2-5)}{\vec{n}_1 \cdot \vec{n}_1} \vec{n}_1 \right\| = \frac{|(3,-1)\cdot(4,-3)|}{\|\vec{n}_1\|} = \frac{15}{\sqrt{10}}$$

Let $n_2 = (3,-1,0)$ in $\mathbb{R}^3$. Then $\vec{n}_2$ is orthogonal to the plane $y=3x+5$. Since $(0,5,0)$ is on the plane, we have

Distance from $(4,2,1)$ to the plane $y = 3x+5$

$$= \left\| \frac{n_2 \cdot (4-0, 2-5, 1-0)}{\vec{n}_2 \cdot \vec{n}_2} \vec{n}_2 \right\| = \frac{|(3,-1,0)\cdot(4,-3,1)|}{\|(3,-1,0)\|} = \frac{15}{\sqrt{10}}$$

   (b) There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the section of the graph of f by the plane $x = 0$ is the set

$$\{(x,y,z) \in \mathbb{R}^3 : x = 0 \text{ and } y^2 + z^2 = 1\}.$$

FALSE. If this were true, then the points $(0,0,1)$ and $(0,0,-1)$ would both be on the graph of f, but this is impossible since f is a function (i.e. it would fail the "vertical line test").

(c) There is a twice continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\frac{\partial f}{\partial x} = e^{x^2} \text{ and } \frac{\partial f}{\partial y} = 2xy.$$

FALSE. If f is C^2, then we know

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

But $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(e^{x^2}) = 0$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(2xy) = 2y$$

(d) If length ℓ of a rectangle is increasing with respect to time at a rate of 1 in/sec and its width is increasing with respect to time at a rate of 2 in/sec, then its area is increasing with respect to time at a rate of 4 in^2/sec at the instant its width is 2 inches and its ~~height~~ length is 1 inch.

TRUE $A = \ell w$, so by the Chain Rule

$$\frac{\partial A}{\partial t} = \begin{bmatrix} \frac{\partial A}{\partial \ell} & \frac{\partial A}{\partial w} \end{bmatrix} \begin{bmatrix} \frac{\partial \ell}{\partial t} \\ \frac{\partial w}{\partial t} \end{bmatrix}$$

$$= \begin{bmatrix} w & \ell \end{bmatrix} \begin{bmatrix} \frac{\partial \ell}{\partial t} \\ \frac{\partial w}{\partial t} \end{bmatrix} = w\frac{\partial \ell}{\partial t} + \ell\frac{\partial w}{\partial t} = w(1) + \ell(2) = w + 2\ell$$

When $w = 2$ and $\ell = 1$

$$\frac{\partial A}{\partial t} = 2 + 2(1) = 4$$

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

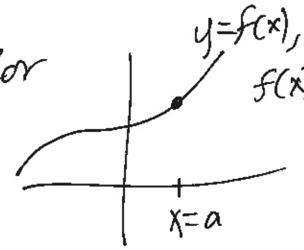
   (a) Say that two lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$ are distance d apart. If PQ is a line segment connecting P on ℓ_1 to Q on ℓ_2 and length$(PQ) = d$, then PQ is perpendicular to both ℓ_1 and ℓ_2.

ALWAYS. Let $\vec{n}$ be ~~the~~ a vector normal to ℓ_1 & ℓ_2.

If $\|\text{Proj}_{\vec{n}}(PQ)\| = \|PQ\| = d$, then PQ points in the same direction as $\vec{n}$ and is thus normal to ℓ_1 & ℓ_2.

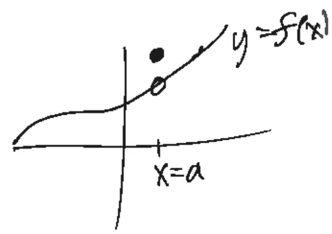
   (b) Let $\vec{a} \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^m$. If $\lim_{\vec{x} \to \vec{a}} f(\vec{x})$ exists, then f is continuous at $\vec{a}$.

SOMETIMES: for



f(x) is continuous & $\lim_{\vec{x} \to \vec{a}} f(x)$ exists;

but, for



$\lim_{\vec{x} \to \vec{a}} f(x) \neq f(\vec{a})$ so f(x) is not continuous at $\vec{a}$.

(c) Suppose $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ are differentiable and $f(\vec{0}) = g(\vec{0}) = \vec{0}$. If $Df(\vec{0})$ is the zero matrix, then $D(f \circ g)(\vec{0})$ is also the zero matrix.

ALWAYS:

$$D(f\circ g)(\vec{0}) = Df(g(\vec{0}))\, Dg(\vec{0})$$
$$= Df(\vec{0})\, Dg(\vec{0}) = \vec{0} \cdot Dg(\vec{0}) = 0$$

(d) The directional derivative of $f(x, y) = x^2 + y^2$ at a non-zero point (a, b) and in the direction determined by the vector (a, b) is 4.

SOMETIMES:

Let $\vec{v} = \frac{(a,b)}{\|(a,b)\|}$; then $D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v} =$

$$= (2a, 2b) \cdot \frac{(a,b)}{\sqrt{a^2+b^2}} = \frac{2a^2+2b^2}{\sqrt{a^2+b^2}} = 2\sqrt{a^2+b^2}.$$

If $(a,b) = (2,0)$ then $D_v f(\vec{a}) = 4$

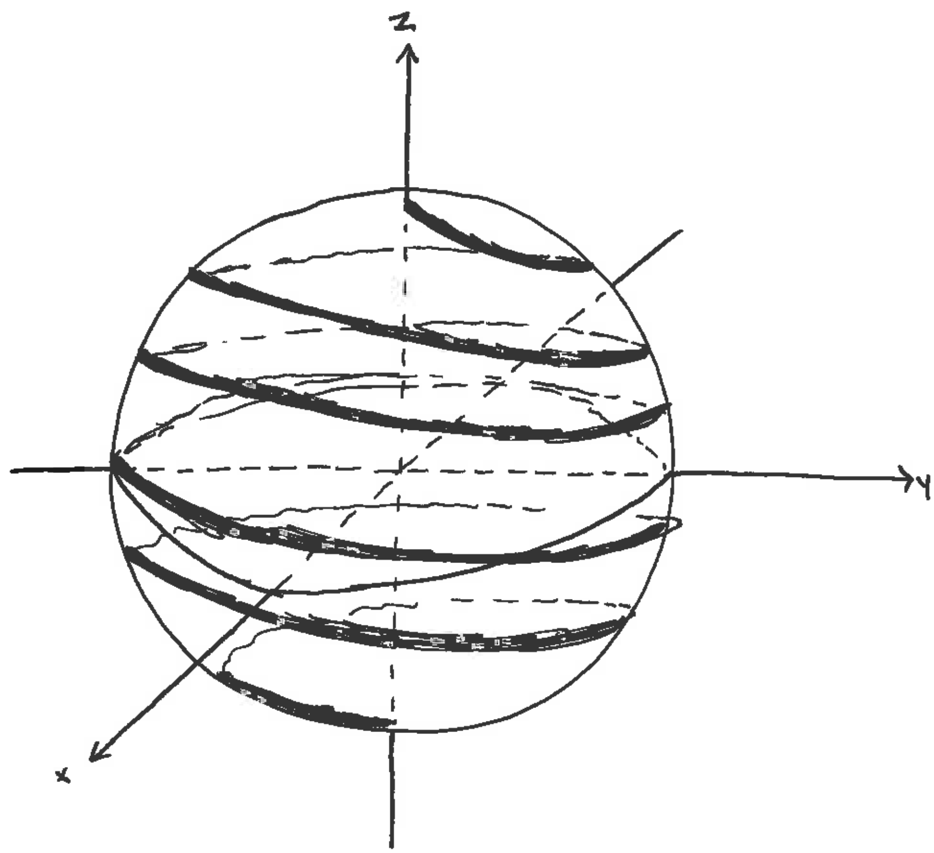
If $(a,b) = (1,0)$ then $D_v f(\vec{a}) = 2 \neq 4$.

3. Describe and sketch the intersection (in $\mathbb{R}^3$) of the surface given by the spherical equation $\rho = 3$, and the surface given by the spherical equation $\phi = (1/10)\theta$, where θ varies between 0 and 10π.

The θ coordinate makes 5 complete revolutions

The ϕ coordinate moves from the north pole to the south pole

ρ is always 3 (so the curve of intersection is on the sphere of radius 3 centered at the origin).



4. For each of the following limits, explain why it does or does not exist, and evaluate it if it does exist.

+9 (a) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+x^2y^2+y^4}$

Along $x=0$: $\lim_{y\to 0} \frac{0}{y^4} = 0$

Along $x=y$: $\lim_{x\to 0} \frac{x^4}{3x^4} = \frac{1}{3}$

So limit does not exist

+6 (b) $\lim_{(x,y,z)\to(0,1,0)} \frac{xyz}{x^2+y^2+z^2}$

xyz and $x^2+y^2+z^2$ are continuous at $(0,1,0)$ and $x^2+y^2+z^2$ is non-zero there,

So $\frac{xyz}{x^2+y^2+z^2}$ continuous at $(0,1,0)$.

Thus limit is $\frac{0\cdot 1\cdot 0}{0^2+1^2+0^2} = 0$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the differentiable function $f(x, y) = \sin(x + y) + \cos(x - y)$.

(a) Find an equation for the tangent plane to the graph $z = f(x, y)$ at the point $(\pi, \pi, 1)$.

Note that $1 = f(\pi, \pi)$.

$f_x = \cos(x+y) - \sin(x-y)$, $f_y = \cos(x+y) + \sin(x-y)$

$f_x(\pi, \pi) = \cos(2\pi) - \sin(0) = 1$, $f_y(\pi, \pi) = \cos(2\pi) + \sin(0) = 1$

The tangent plane is given by

$$P(x,y) = f(\pi,\pi) + f_x(\pi,\pi)(x-\pi) + f_y(\pi,\pi)(y-\pi)$$
$$= 1 + 1(x-\pi) + 1(y-\pi).$$

(b) Use linear approximation at the point $(a, b) = (\pi, \pi)$ to estimate the value of $f(\pi + 0.1, \pi - 0.2)$.

Just plug into the equation for the tangent plane:

$$f(\pi+0.1, \pi-0.2) \approx P(\pi+0.1, \pi-0.2)$$
$$= 1 + (\pi+0.1) - \pi + (\pi-0.2) - \pi = 0.9$$

6. Find an equation of the tangent plane to the surface $z^2 + x^3y - zxy = 1$ at $(1, 1, 1)$.

The plane is described by

$$\nabla f(1,1,1) \cdot (x-1, y-1, z-1) = 0.$$

$$\nabla f = (3x^2y - zy,\ x^3 - zx,\ 2z - xy)$$

So

$$\nabla f(1,1,1) = (2, 0, 1)$$

and the plane is

$$2(x-1) + (z-1) = 0 \quad \text{or}$$

$$2x + z = 3$$

7. You're out for a jog when a tiny volcano near you erupts. Suppose the volcano is at the origin, and if x and y are measured in meters, then the temperature (in hundreds of degrees celsius) at the point (x, y) near the volcano is given by

$$T(x, y) = 10 - \ln(x^4 + y^4 + 1)$$

Suppose you are at the point $(2, 3)$ when the volcano erupts.

+6 (a) Suppose we run away in a straight line at a speed of 6 meters/second, in the direction of the vector $(1, 1)$, so that $\frac{dx}{dt} = 3\sqrt{2}$ and $\frac{dy}{dt} = 3\sqrt{2}$. When we set off, what will the rate of change of T with respect to time be?

$$\frac{\partial T}{\partial x} = -\frac{4x^3}{x^4+y^4+1} \qquad \frac{\partial T}{\partial y} = -\frac{4y^3}{x^4+y^4+1}$$

$$T_x(2,3) = -\frac{32}{98} = \frac{-16}{49} \qquad T_y(2,3) = -\frac{108}{98} = -\frac{54}{49}$$

$$\frac{\partial T}{\partial t}(2,3) = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = \frac{-16}{49}(3\sqrt{2}) - \frac{54}{49}(3\sqrt{2})$$

$$= \frac{-70}{49}(3\sqrt{2}) = -\frac{10}{7}(3\sqrt{2})$$

+4 (b) Find any vector $\vec{u}$ that will make T decrease most rapidly if we run away in the direction of $\vec{u}$.

$$-\nabla T(2,3) = \left(\frac{16}{49}, \frac{54}{49}\right)$$

or a positive multiple of this