1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer.

(a) The function $f(x,y) = e^{xy} - \sin(x^2 + y)$ is differentiable at $(0,0)$.

TRUE

$f_x = ye^{xy} - 2x\cos(x^2+y)$

$f_y = xe^{xy} - \cos(x^2+y)$

Each is a sum of products of compositions of continuous fcns, hence continuous.

So f is differentiable everywhere.

(b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy $f(1,1) = g(1,1) = (-1,-1)$, $Df(x,y) = \begin{bmatrix} x & -2y \\ -2x & 3y \end{bmatrix}$, and $Dg(x,y) = \begin{bmatrix} 2x & y \\ x & x \end{bmatrix}$. Then $D(f \circ g)(1,1) = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$.

Note that f and g are differentiable.

FALSE

$$D(f \circ g)(1,1) = Df(g(1,1))\,Dg(1,1)$$

$$= \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

(c) If P_1 and P_2 are the planes given by $x - y + 3z = 2$ and $-2x + y - 2z = -2$ respectively, then there is exactly one plane that contains both the intersection of P_1 and P_2 and the point $(1, 1, 1)$.

TRUE Note that $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ are not parallel, so P_1 and P_2 are not parallel, so they have a line of intersection.

$1 - 1 + 3 = 3 \neq 2$

so $(1,1,1)$ is not on this line.

So there is a unique plane containing the line and $(1,1,1)$.

(d) Let $f(x, y, z) = x^{2093}y^{84}z - 2x^2y^{53}z^{17}$. Then $f_{xzxzx}(1, -1, 2) = 0$.

TRUE f is a polynomial, so it has continuous partials of all orders.

Hence, by Clairaut's,

$$\frac{\partial^5}{\partial x \partial z \partial x \partial z \partial x}(x^{2093}y^{84}z) = \frac{\partial^5}{\partial x^3 \partial z^2}(x^{2093}y^{84}z) = 0$$

and

$$\frac{\partial^5}{\partial x \partial z \partial x \partial z \partial x}(2x^2y^{53}z^{17}) = \frac{\partial^5}{\partial z^2 \partial x^3}(2x^2y^{53}z^{17}) = 0$$

so $f_{xzxzx}(1,-1,2) = 0$.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer

(a) For a number k, the level surfaces of $g(x,y,z) = (k^2-k)\sin x + 3x - y + z - (k^2-1)z^3$ are planes.

SOMETIMES

we need $(k^2-k)\sin x + 3x - y + z - (k^2-1)z^3$
$= ax + by + cz + d$

for some $a,b,c,d \in \mathbb{R}$.

So we need $k^2 - k = 0 \Rightarrow k(k-1) = 0 \Rightarrow k=0$ or $k=1$
$k^2 - 1 = 0 \Rightarrow k = \pm 1$

If $k=1$, the level surfaces are all planes
If not, they aren't.

(b) For a real number $0 \le k \le \pi$, the equation $\varphi = k$ in spherical coordinates represents a plane.

SOMETIMES

If $k = \frac{\pi}{2}$, then it represents the xy-plane

If $k=0$, it represents the positive z-axis.

(c) For a real number k, the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2+k(x-1)y}{\sqrt{x^2+y^2}}$ exists.

SOMETIMES

If $k=0$, $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$

If $k=1$, we have $\dfrac{x^2+y^2+(x-1)y}{\sqrt{x^2+y^2}}$.

Let $y=0$. Then $\dfrac{x^2}{\sqrt{x^2}} = |x| \to 0$

Let $x=y$. Then $\dfrac{2x^2+x^2-x}{\sqrt{2x^2}} = \dfrac{3x^2-x}{\sqrt{2x^2}}$

$\left|\dfrac{3x^2-x}{\sqrt{2x^2}}\right| = \left|\dfrac{3}{\sqrt{2}}|x| - \dfrac{x}{|x|}\right| \to 1 \neq 0$. Limit d.n.e.

(d) Given $\vec{a}, \vec{b} \in \mathbb{R}^3$ with $\vec{a} \neq \vec{0}$, we have $\vec{a} \times \vec{b} = \vec{a}$.

FALSE

$(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ since $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$

so if $\vec{a} \times \vec{b} = \vec{a}$ then we have

$0 = \vec{a} \cdot \vec{a} = \|\vec{a}\|^2 \Rightarrow \|\vec{a}\| = 0$. But we've assumed $\vec{a} \neq \vec{0}$.

3. Let L_1 and L_2 be the (skew) lines given by the parameterizations $\vec{r}_1(t) = \begin{bmatrix} 3+2t \\ 1+2t \\ t \end{bmatrix}$ and $\vec{r}_2(t) = \begin{bmatrix} 1+2t \\ 2+3t \\ -1+2t \end{bmatrix}$ respectively. Find the distance between L_1 and L_2.

The direction vectors for L_1 and L_2 are

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \text{ respectively.}$$

These are not parallel so there is a unique plane parallel to both.

Let $\vec{n} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix}\vec{k}$

$$= 1\vec{i} - 2\vec{j} + 2\vec{k} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

Let P_1 be given by $1(x-3) - 2(y-1) + 2(z-0) = 0$

and P_2 by $1(x-1) - 2(y-2) + 2(z+1) = 0$

These are parallel planes containing L_1 and L_2 respectively.

The vector $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ connects the two planes.

Projecting onto $\vec{n}$ gives the distance between P_1 and P_2, which is the distance between L_1 and L_2.

$$\text{proj}_{\vec{n}} \vec{v} = \frac{\vec{n} \cdot \vec{v}}{\|\vec{n}\|^2}\vec{n} = \frac{6}{1+4+4}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

$$\text{Distance} = \left\| \frac{2}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\| = \frac{2}{3}\sqrt{(1+4+4)} = \frac{2}{3}\sqrt{9} = 2.$$

4. An ant is walking on an anthill, which has altitude $A(x, y) = 15 - 3x^2 - 2y^2$ inches, where x is the number of inches east of the peak, and y is the number of inches north of the peak.

(a) Suppose the ant is currently 1 inch west of the peak and 2 inches north of it. If he moves due east, what will be the slope of his path initially?

$A_x = -6x$

$A_x(-1,2) = 6$

(b) If the ant instead moves due north, what will the slope of his path be initially?

$A_y = -4y$

$A_y(-1,2) = -8$

(c) What is the slope of the ants path initially if he moves directly towards the peak, i.e. in the direction of $\left[\begin{smallmatrix}1\\-2\end{smallmatrix}\right]$?

$\vec{u} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}$ is a unit vector in the direction of $\begin{bmatrix}1\\-2\end{bmatrix}$.

So since A is differentiable (it's a polynomial),

the slope is $D_{\vec{u}}A(1,-2) = \left(\frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}\right) \cdot \nabla A(-1,2) = \frac{1}{\sqrt{5}}\left(\begin{bmatrix}1\\-2\end{bmatrix} \cdot \begin{bmatrix}6\\-8\end{bmatrix}\right)$

$= \frac{1}{\sqrt{5}}(6 + 16) = \frac{22}{\sqrt{5}}$

5. Find the linear approximation of the function $f(x, y, z) = (xe^{yz}, \sin(x^2y) + z)$ at $(1, 0, 1)$.

Let $f_1 = xe^{yz}$ $\quad f_2 = \sin(x^2y) + z$

Then $Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}$

$= \begin{bmatrix} e^{yz} & xze^{yz} & xye^{yz} \\ 2xy\cos(x^2y) & x^2\cos(x^2y) & 1 \end{bmatrix}$

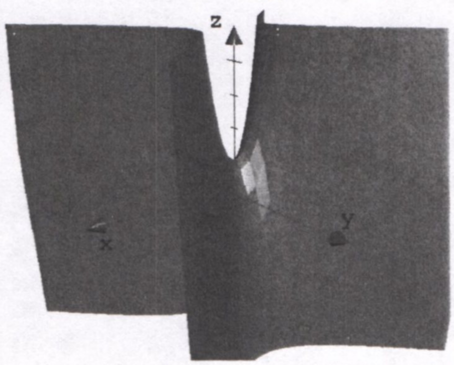
$L(x,y,z) = f(1,0,1) + Df(1,0,1)\begin{bmatrix} x-1 \\ y-0 \\ z-1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x-1 \\ y-0 \\ z-1 \end{bmatrix}$

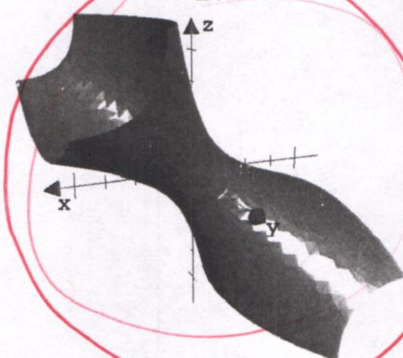
$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} x-1+y \\ y+z-1 \end{bmatrix} = \begin{bmatrix} x+y \\ y+z \end{bmatrix}.$

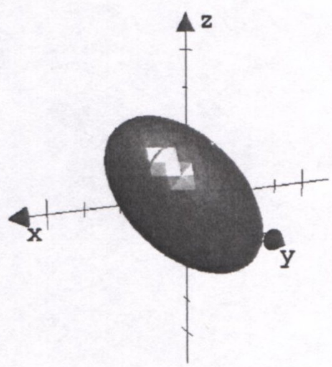
6. Which of the following surfaces is given by $x^2 + y^2 + z^2 - 4xz = 4$. **Justify your answer below. Hint: Use linear algebra to write an equation for the surface that doesn't have any mixed terms.**

A.

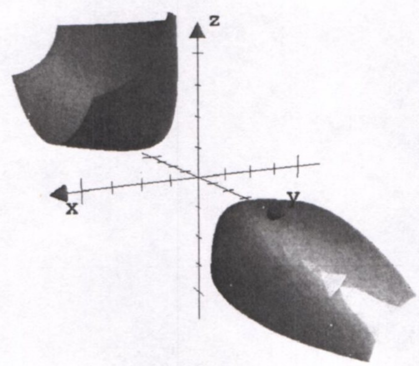


B.





C.



D.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad \det \begin{bmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0 - 2\begin{vmatrix} 0 & 1-\lambda \\ -2 & 0 \end{vmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2\right] - 4(1-\lambda)$$

$$= (1-\lambda)(\lambda^2 - 2\lambda + 1 - 4) = (1-\lambda)\left[\lambda^2 - 2\lambda - 3\right]$$

$$= (1-\lambda)(\lambda - 3)(\lambda + 1)$$

$$c_1^2 + 3c_2^2 - c_3^2 = 4$$

$c_3 = k \rightarrow$ ellipse, defined for all k

$c_1 = 0 \rightarrow$ hyperbolas

$c_2 = 0 \rightarrow$ hyperbolas