1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **four** parts.)

   (a) The graph of $\rho = 1 - \sin(\phi)$ describes a sphere.

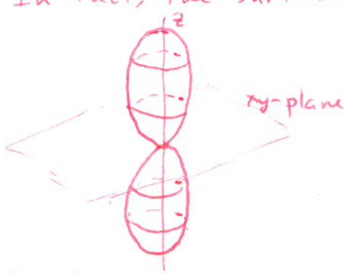
Answer: False

θ is not in the equation, so all θ-cross sections are the same, so this <u>is</u> a surface of revolution.

However, it contains 3 collinear points:
so it cannot be a sphere.

ϕ	$\sin\phi$	$\rho = 1 - \sin\phi$	(x,y,z)
0	0	1	$(0,0,1)$
$\pi/2$	1	0	$(0,0,0)$
π	0	1	$(0,0,-1)$

In fact, the surface looks like:



   (b) There exist numbers k and ℓ such that level sets of the functions $f(x,y,z) = x+y+z$ and $g(x,y,z) = x+y+z+1$ at levels k and ℓ, respectively, are the same surface.

Answer: True

Let $k=0 \Rightarrow f(x,y,z) = k = 0 = x+y+z$

and $\ell = 1 \Rightarrow g(x,y,z) = \ell = 1 = x+y+z+1$

$$\begin{array}{rcl} 1 &=& x+y+z+1 \\ -1 & & -1 \\ \hline 0 &=& x+y+z \end{array}$$

these are the same surface!

In fact, if $f(x,y,z) = k = x+y+z$

then $k+1 = x+y+z+1 = g(x,y,z)$

so for $\ell = k+1$, the surfaces $f(x,y,z) = k$ and $g(x,y,z) = \ell$ describe the same surface.

(c) There is a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x, y) = xy = \frac{\partial f}{\partial y}(x, y).$$

Answer: FALSE

If there were a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x,y) = xy = \frac{\partial f}{\partial y}(x,y),$$

then by Clairaut's Theorem,

$$x = \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) = y$$

which is not true for some $(x,y) \in \mathbb{R}^2$.

(d) There is a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ such that the directional derivative $D_{\mathbf{u}}f(\mathbf{0}) > 0$ for every unit vector $\mathbf{u} \in \mathbb{R}^n$.

Answer: FALSE

If there were a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $D_{\vec{u}} f(\vec{0}) > 0$ for each unit vector $\vec{u} \in \mathbb{R}^n$, then for each unit vector $\vec{u} \in \mathbb{R}^n$, $-\vec{u}$ is a unit vector, and

$$\begin{aligned} D_{-\vec{u}} f(\vec{0}) &= \nabla f(\vec{0}) \cdot (-\vec{u}) \\ &= -\nabla f(\vec{0}) \cdot \vec{u} \\ &= -D_{\vec{u}} f(\vec{0}) \\ &< 0 \end{aligned}$$

This contradicts $D_{-\vec{u}} f(\vec{0}) > 0$.

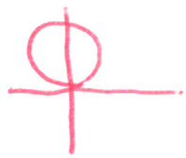
2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **four** parts.)

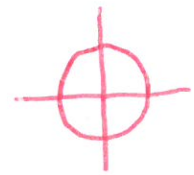
   (a) For a function $f(\theta)$, the polar graphs of $r = f(\theta)$ and $r = f(-\theta)$ are different.

   Answer: Sometimes

   True if $f(\theta) = \sin(\theta)$ False if $f(\theta) = 1$

   $r = \sin(\theta)$ $r = \sin(-\theta) = -\sin\theta$ $r = 1$ (independent of θ)





   (b) For $a \geq 0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|^a y^3 + x^2 y}{x^2 + y^2}$ exists.

   Answer: Always

$$\lim_{(x,y)\to(0,0)} \frac{|x|^a y^3 + x^2 y}{x^2 + y^2} = \lim_{r\to 0} \frac{r^{\alpha+3}|\cos\theta|\sin^3\theta + r^3\cos^2\theta\sin\theta}{r^2}$$

$$= \lim_{r\to 0} \underbrace{r^{\alpha+1}}_{\to 0 \text{ if } \alpha+1>0}\underbrace{|\cos\theta|\sin^3\theta}_{\text{bounded}} + \underbrace{r}_{\to 0}\underbrace{\cos^2\theta\sin\theta}_{\text{bounded}}$$

$= 0$ as long as $\alpha > -1$,

so always for $\alpha \geq 0$.

(c) For a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ where $x = x(t)$ and $y = y(t)$ are each twice-differentiable functions of a variable t,

$$\frac{d^2 f}{dt^2} = \frac{\partial^2 f}{\partial x^2}\frac{d^2 x}{dt^2} + \frac{\partial^2 f}{\partial y^2}\frac{d^2 y}{dt^2}.$$

Answer: SOMETIMES

True example: $f(x,y)=0 \quad x=0 \quad y=0$

then both sides are 0

False example: $f(x,y)=xy \quad x=t \quad y=t$

$f(x(t),y(t))=t^2$

so $\dfrac{d^2f}{dt^2}=2 \neq 0\cdot 0+0\cdot 0$

(d) For a point (a,b) in $\mathbb{R}^2$, the tangent plane to the sphere $x^2+y^2+z^2=1$ at $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is parallel to the tangent plane to the graph of $f(x,y) = -xy^2 - x + 2y$ at $(a,b,f(a,b))$.

Answer: NEVER

tangent plane to graph f : $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

so normal $= (-f_x(a,b), -f_y(a,b), 1)$

$= (b^2+1,\ 2ab-2,\ 1)$

normal to sphere $= \nabla$ of $g(x,y,z) = x^2+y^2+z^2$

$= (2(1/\sqrt{3}), 2(1/\sqrt{3}), 2(1/\sqrt{3}))$

For normals to be parellel need

$b^2+1 = 2ab-2 = 1$, but $b^2+1=1 \Rightarrow b=0$

and then $2a(0)-2 \neq 1$ for all a

3. Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ given by $F(x, y, z) = x^2 + y^2 + 2\sqrt{2}xz$. For which numbers k does $F(x, y, z) = k$ describe a two-sheeted hyperboloid?

Change coordinates so that the ~~equation~~ quadratic form is in a more familiar form:

This amounts to diagonalizing $A = \begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$.

Solve: $$0 = \det(A - \lambda I)$$
$$= \det\begin{pmatrix} 1-\lambda & 0 & \sqrt{2} \\ 0 & 1-\lambda & 0 \\ \sqrt{2} & 0 & -\lambda \end{pmatrix}$$
$$= (1-\lambda)\begin{vmatrix} 1-\lambda & \sqrt{2} \\ \sqrt{2} & -\lambda \end{vmatrix}$$
$$= (1-\lambda)\left[(1-\lambda)(-\lambda) - 2\right]$$
$$= (1-\lambda)(\lambda-2)(\lambda+1)$$

$\lambda = 2, \pm 1$

In new coordinates, $F(c_1, c_2, c_3) = c_1^2 + 2c_2^2 - c_3^2 = k$
$$c_1^2 + 2c_2^2 = k + c_3^2$$

$\Rightarrow \boxed{k < 0}$ for a 2-sheeted hyperboloid

4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$f(x,y,z) = \begin{cases} (x^2+y^2+z^2)\sin\left(\frac{1}{x^2+y^2+z^2}\right) & (x,y,z) \neq (0,0,0) \\ k & (x,y,z) = (0,0,0). \end{cases}$$

Find a value of k which makes f continuous at $(0,0,0)$.

If $f(0,0,0) = \lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$, then f is continuous at $(0,0,0)$.

Want: $k = \lim_{(x,y,z)\to(0,0,0)} (x^2+y^2+z^2)\sin\left(\frac{1}{x^2+y^2+z^2}\right)$

$= \lim_{\rho\to 0^+} \rho^2 \sin\left(\frac{1}{\rho^2}\right)$ convert to spherical

$\forall \rho > 0$: $-\rho^2 \leq \rho^2 \sin\left(\frac{1}{\rho^2}\right) \leq \rho^2$

$\lim_{\rho\to 0^+} -\rho^2 \leq \lim_{\rho\to 0^+} \rho^2 \sin\left(\frac{1}{\rho^2}\right) \leq \lim_{\rho\to 0^+} \rho^2$

$0 \leq \lim_{\rho\to 0^+} \rho^2 \sin\left(\frac{1}{\rho^2}\right) \leq 0$

OR: It is enough to say that sine is bounded between -1 & 1 and $\rho^2 \to 0$ as $\rho \to 0$.

$\Rightarrow$ Squeeze $k = \lim_{\rho\to 0^+} \rho^2 \sin\left(\frac{1}{\rho^2}\right) = 0$

5. Find a linear approximation to the function $\mathbf{g}(x, y, z) = (2^{x+y+z}, \sin(x+y-2z))$ at $(1, 1, 1)$ and use it to approximate $\mathbf{g}(1, 0.9, 1.1)$. (Recall that the derivative of $f(x) = 2^x$ with respect to x is $2^x \ln 2$.)

15 points

Let $g = (f, h)$.

Then, $f_x = 2^{x+y+z} \ln 2$

$f_y = 2^{x+y+z} \ln 2$

$f_z = 2^{x+y+z} \ln 2$

$h_x = \cos(x+y-2z)$

$h_y = \cos(x+y-2z)$

$h_z = -2\cos(x+y-2z)$

$$Df(x,y,z) = \begin{bmatrix} 2^{x+y+z} \ln 2 & 2^{x+y+z} \ln 2 & 2^{x+y+z} \ln 2 \\ \cos(x+y-2z) & \cos(x+y-2z) & -2\cos(x+y-2z) \end{bmatrix}$$

Best linear approximation at $(1,1,1)$:

$$g(1,1,1) + Df(1,1,1)\begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}$$

$$= (8, 0) + \begin{bmatrix} 8\ln 2 & 8\ln 2 & 8\ln 2 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}$$

$$= (8, 0) + (8\ln 2(x+y+z-3), x+y-2z)$$

$$= (8 + 8(\ln 2)(x+y+z-3), x+y-2z)$$

$$g(1, 0.9, 1.1) \approx (8 + 8\ln 2(1+0.9+1.1-3), 1+0.9-2.2)$$

$$= (8 + 8(\ln 2)(0), -0.3)$$

$$= (8, -0.3)$$

6. (This problem has **two** parts.) The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \frac{1}{69\pi} \sin\left((3x^2 + 5y^2)\pi\right) - 7$$

describes the air temperature in degrees Celsius on a patch of ice at position (x, y). Wally the Walrus is wallowing in some snow at position $(2, -1)$.

(a) In which direction should Wally waddle to warm up most quickly? Give your answer as a (not necessarily unit) vector.

$$\cancel{\partial f} \quad f_x = \frac{6x\pi \cos((3x^2+5y^2)\pi)}{69\pi}$$

$$= \frac{2x}{23} \cos((3x^2+5y^2)\pi)$$

$$f_y = \frac{10y\pi \cos((3x^2+5y^2)\pi)}{69\pi}$$

$$= \frac{10y}{69} \cos((3x^2+5y^2)\pi)$$

$$\nabla f(2,-1) = \left(\frac{4}{23} \cos(17\pi), \frac{-10}{69} \cos(7\pi) \right)$$

$$= \left(\frac{-4}{23}, \frac{10}{69} \right).$$

Warm up quickest in direction of $\left(\frac{-4}{23}, \frac{10}{69} \right)$.

(b) At some time, Wally waddles through the point $(3, 2)$ following the curve with parametric equations

$$(x(t), y(t)) = (t + 2, 3t^2 - 1),$$

where t is measured in hours. What is the rate of change in air temperature with respect to time that Wally experiences as he waddles through the position $(3, 2)$? The air temperature is described by the same function $f(x, y) = \frac{1}{69\pi}\sin\left((3x^2 + 5y^2)\pi\right) - 7$ as before.

Note that Wally waddles through $(3,2)$ when $t=1$.

Method 1: Chain Rule

$$\frac{df}{dt}(1) = \frac{\partial f}{\partial x}(3,2)\frac{dx}{dt}(1) + \frac{\partial f}{\partial y}(3,2)\frac{dy}{dt}(1)$$

$$= \frac{6(3)\pi}{69\pi}\cos(47\pi)(1) + \frac{10(2)\pi}{69\pi}\cos(47\pi)(6\cdot 1)$$

$$= \frac{-18}{69} - \frac{120}{69} = \frac{-138}{69} = \boxed{-2\ {}^\circ/hr}$$

Method 2: Substitution

$$f(t) = \frac{1}{69\pi}\sin\left(\left(3(t+2)^2 + 5(3t^2-1)^2\right)\pi\right) - 7$$

$$= \frac{1}{69\pi}\sin\left(\left(45t^4 - 27t^2 + 12t + 17\right)\pi\right) - 7$$

$$f'(t) = \frac{1}{69}\left(180t^3 - 54t + 12\right)\cos\left(\left(45t^4 - 27t^2 + 12t + 17\right)\pi\right)$$

$$f'(1) = \frac{1}{69}(138)\cos(47\pi) = \frac{-138}{69} = \boxed{-2\ {}^\circ/hr}$$