Notes on Green's Theorem
Northwestern, Spring 2013

The purpose of these notes is to outline some interesting uses of Green's Theorem in situations where it doesn't seem like Green's Theorem should be applicable. Such applications aren't really mentioned in our book, and I consider this to be a travesty. The power of the various theorems we will be learning about in Vector Calculus rests in their ability to be applied in a wide range of scenarios.

At the end I also give some insight into the intuition behind Green's Theorem, to convince you that it is a pretty obvious result when you interpret it in the right way. Actually, this is a lie: it is not at all obvious, but at least I hope to give you an idea as to how someone might guess that such a thing should be true in the first place.

Closing Off a Curve

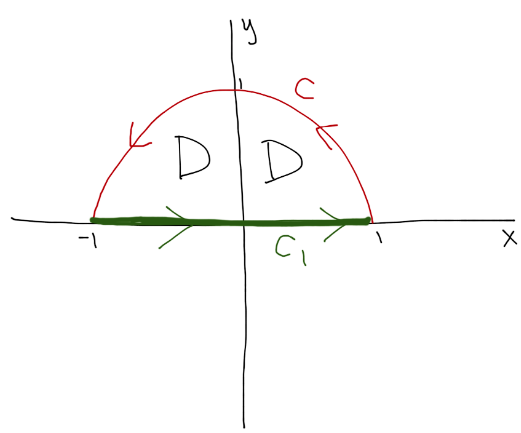
In the statement of Green's Theorem, the curve we are integrating over should be closed and oriented in a way so that the region it is the boundary of is on its left, which usually (though not always, as we shall see) means the curve should be oriented counterclockwise. Having the wrong orientation is easy to deal with: we simply change the sign of the resulting double integral in Green's Theorem:

$$\int_{\substack{C \text{ with wrong} \\ \text{orientation}}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\substack{C \text{ with correct} \\ \text{orientation}}} \mathbf{F} \cdot d\mathbf{s} = -\iint_D (\operatorname{curl} \mathbf{F} \cdot \mathbf{k}) \, dA.$$

However, how do we deal with having a curve which is not closed? The answer is: we "close off" the curve, apply Green's Theorem, and then subtract the integral over the piece with glued on. Here is an example to illustrate this idea:

Example 1. Consider the line integral of $\mathbf{F} = (y^2x + x^2)\mathbf{i} + (x^2y + x - y^{y \sin y})\mathbf{j}$ over the top-half of the unit circle C oriented counterclockwise. Clearly, this line integral is going to be pretty much impossible to compute directly if we parametrize C due to the $y^{y \sin y}$ term in $\mathbf{F}$. And, since C is not closed at first glance it seems that Green's Theorem is not applicable. The technique of "Closing off" the curve comes to the rescue!

We close off C by gluing on the line segment C_1 from $(-1, 0)$ to $(1, 0)$; denote the resulting closed curve by $C + C_1$:



Note that with the orientation we gave C_1 (i.e. the rightward one) the closed curve $C + C_1$ has the correct orientation needed to be able to apply Green's Theorem. We now use the fact that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C+C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_1} \mathbf{F} \cdot d\mathbf{s}.$$

We can compute the first line integral on the right using Green's Theorem, and the second one will be much simpler to compute directly than the original one due to the fact that C_1 is an easy curve to deal with.

Letting D denote the region enclosed by $C + C_1$, we have

$$\int_{C+C_1} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\operatorname{curl} \mathbf{F} \cdot \mathbf{k})\, dA = \iint_D (2xy + 1 - 2yx)\, dA = \frac{\pi}{2}.$$

Parametrizing C_1 as $\mathbf{x}(t) = (t, 0)$ with $-1 \le t \le 1$, we get

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{-1}^{1} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt = \int_{-1}^{1} t^2\, dt = \frac{2}{3}.$$

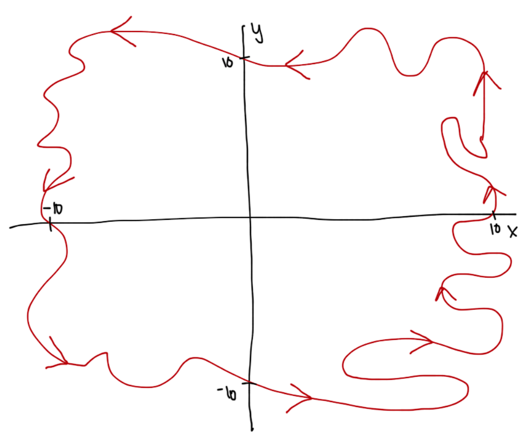
Thus the line integral we want is

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C+C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \frac{\pi}{2} - \frac{2}{3}.$$

To recap: we closed off the curve C, applied Green's Theorem to the result, and then subtracted off the piece we glued on.

Replacing a Closed Curve

Often times we are integrating over a closed curve but Green's Theorem isn't applicable due to the vector field not being defined throughout the entire region enclosed by the curve. Consider for instance the vector field $\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2}$ and the closed curve C drawn below:

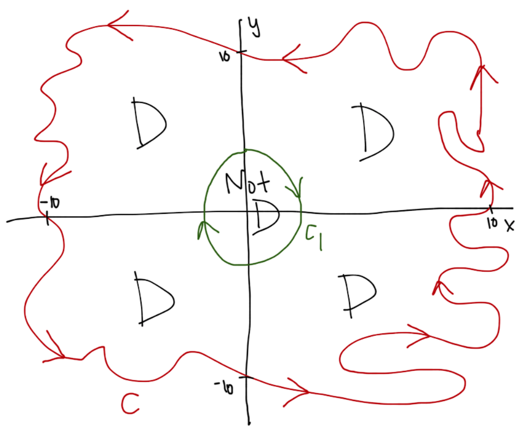


First, it is pretty much impossible to compute the line integral of $\mathbf{F}$ over C directly since we can't really write down parametric equations for C. Our next guess would be to use Green's Theorem, but this is not possible since $\mathbf{F}$ isn't defined at the origin which is in the region enclosed by C. So, what's left? The answer is to replace C by a simpler closed curve C_1 which will give the same line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s}.$$

In this case, let C_1 be the unit circle with orientation to be decided later. The claim is that for a certain orientation on C_1, the above two line integrals will in fact be equal. This is good, since the line integral over C_1 is straightforward to compute directly: we simply use the standard parametric equations for the unit circle in terms of $\cos t$ and $\sin t$. Note that with these equations the denominator of $\mathbf{F}$ simply becomes 1, which is part of the reason why the line integral over C_1 is possible to compute directly.

So the question remains: how do we know that integrating $\mathbf{F}$ over C_1 (with a correct orientation) will give the same value as integrating $\mathbf{F}$ over C? Green's Theorem to the rescue! The key is to consider the "combined" curve $C + C_1$, which consists of two pieces. This might seem strange, but all that we are doing works over such combined curves as well, where by definition to integrate over a combined curve just means that we integrate over each separately and add the results. Now, what is the region "enclosed" by $C + C_1$. It is actually the region D *between* the two curves:



and this region does NOT include the origin! Hence, Green's Theorem is applicable to this region since $\mathbf{F}$ is indeed defined throughout the entire region bounded by $C + C_1$. The only thing which remains is to determine the correct orientation on C_1 so that Green's Theorem applies, which we do in the example below:

Example 2. We want to compute the line integral of $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$ over the curve C drawn above. Let C_1 be the unit circle oriented *clockwise* and consider the combined curve $C + C_1$. Let D be the region between the two curves. Note that the clockwise orientation on C_1 is compatible (i.e. is the correct one) with the one on C since when traveling along either curve the region D will be on your *left*: this is precisely the orientation on a curve needed so that Green's Theorem applies. Since $\operatorname{curl} \mathbf{F} = \mathbf{0}$ in this case (compute it!), we have

$$\int_{C+C_1} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\operatorname{curl} \mathbf{F} \cdot \mathbf{k})\, dA = \iiint_D 0\, dA = 0,$$

and thus

$$\int_C \mathbf{F} \cdot d\mathbf{s} + \int_{C_1} \mathbf{F} \cdot d\mathbf{s} = 0, \text{ so } \int_C \mathbf{F} \cdot d\mathbf{s} = -\int_{C_1} \mathbf{F} \cdot d\mathbf{s}.$$

This final integral is the same as the one over C_1 oriented *counterclockwise*, so we finally conclude that the line integral we want equals the line integral of $\mathbf{F}$ over the unit circle with counterclockwise orientation. We have replaced C by a simpler curve which gives the same line integral. This line integral is now straightforward to compute using the standard parametric equations

$$\mathbf{x}(t) = (\cos t, \sin t),\ 0 \leq t \leq 2\pi$$

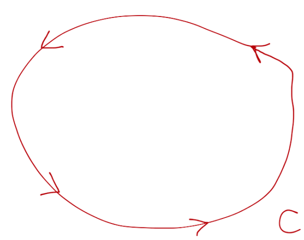
for the unit circle oriented counterclockwise. The result is 2π, so

$$\int_C \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2} \cdot d\mathbf{s} = 2\pi,$$

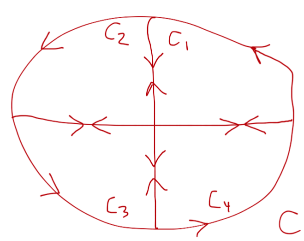
a computation which seemed impossible to carry out at the beginning. To recap: often times we can replace a curve by a simpler curve and still get the same line integral, by applying Green's Theorem to the region between the two curves.

Intuition Behind Green's Theorem

Finally, we look at the reason as to why Green's Theorem makes sense. Consider a vector field $\mathbf{F}$ and a closed curve C:



Consider the following curves C_1, C_2, C_3, and C_4 filling up part of the region enclosed by C:



So, C_1 is the closed curve in the upper-right corner of this region and shares its upper boundary with a part of C, C_2 is the closed curve in the upper-left corner of this region and shares its upper boundary with a part of C, and so on.

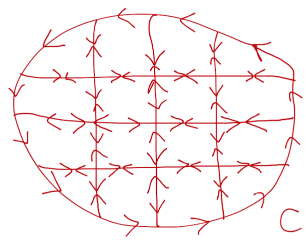
Now, consider what happens when you take the line integral of $\mathbf{F}$ over each of these curves and add up the results. The line integral over C_1 alone can be broken up into the line integrals over each of its three boundary pieces, and similarly for the other curves. However, note that the left boundary piece of C_1 is the same as the right boundary piece of C_2 except with *opposite* orientation! Similarly, the bottom boundary of C_1 is the same as the upper boundary of C_4 except with opposite orientation. The point is that the line integral of $\mathbf{F}$ over all of these interior boundary pieces will cancel out since each will contribute two parts with opposite orientation to the sum

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \int_{C_3} \mathbf{F} \cdot d\mathbf{s} + \int_{C_4} \mathbf{F} \cdot d\mathbf{s}.$$

Thus, all that is left in the above sum is the line integral over the "outer" boundary pieces of C_1, C_2, C_3, and C_4, but these outer boundaries together form all of the original C, so we get that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \int_{C_3} \mathbf{F} \cdot d\mathbf{s} + \int_{C_4} \mathbf{F} \cdot d\mathbf{s} = \int_{C} \mathbf{F} \cdot d\mathbf{s}.$$

Now fill in the region with even more curves:



and do the same thing. The same reasoning will show that adding up all the line integrals over the smaller curves will give the same thing as the line integral over the original curve, and so on and so forth no matter how many times we keep putting in more and more curves:

$$\int_{C} \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{n} \int_{C_i} \mathbf{F} \cdot d\mathbf{s}.$$

Now take a limit as the number of curves n goes to infinity, and as the curves get smaller. As n gets larger, we are looking at the line integrals over curves which are zeroing in on single points:

As curves get smaller, we are measuring circulation around points.

Adding all these circulations up gives a double integral.

Such line integrals provide a better and better measurement of the "circulation" of $\mathbf{F}$ around that point. So, in the limit we end up taking the circulation around each point in D, the region enclosed by C, and adding all of these up measurements up. Adding up quantities at each point of D is what a double integral does, so we get:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{C_i} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\text{circulation around points})\, dA.$$

This circulation is precisely what $\operatorname{curl} \mathbf{F}$ measures! More precisely, in this two-dimensional case where $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ we have

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}.$$

This coefficient of $\mathbf{k}$ is what gives a measurement of the circulation of $\mathbf{F}$ around a given point: a positive value at a point means $\mathbf{F}$ has a counterclockwise circulation around that point and a negative value means it has a clockwise circulation. How positive or how negative the value is tells us how strong the circulation of $\mathbf{F}$ is. Thus, we end up with

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA,$$

which is Green's Theorem. Usually, using the computation of $\operatorname{curl} \mathbf{F}$ above, I like to write this as

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D (\operatorname{curl} \mathbf{F} \cdot \mathbf{k})\, dA,$$

which makes it look more like a fact we will soon discuss: Stokes' Theorem. (This is no accident, as we shall see.) In conclusion, Green's Theorem is really about what happens when you slice through a region with more and more curves; in a sense, *all* of calculus is based on a similar idea.