How to Compute Line Integrals
Northwestern, Spring 2013

Computing line integrals can be a tricky business. First, there's two types of line integrals: scalar line integrals and vector line integrals. Although these are related, we compute them in different ways. Also, there's two theorems flying around, Green's Theorem and the Fundamental Theorem of Line Integrals (as I've called it), which give various other ways of computing line integrals. Knowing what to use when and where and why can be confusing, so hopefully this will help to keep things straight.

I won't say much about what line integrals mean geometrically or otherwise. Hopefully this is something you have some idea about from class or the book, but I'm always happy to talk about this more in office hours. Here the focus is on computation.

Key Point: When you are trying to apply one of these techniques, make sure the technique you are using is actually applicable! (This was a major source of confusion on Quiz 5.)

Scalar Line Integral

- Arises when integrating a *function* f over a curve C.
- Notationally, is always denoted by something like

$$\int_C f\,ds,$$

  where the s in ds is just a normal s, as opposed to $d\mathbf{s}$ or $d\vec{s}$.

- To compute, find parametric equations $\mathbf{x}(t), a \leq t \leq b$ for C and use

$$\int_C f\,ds = \int_a^b f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt.$$

- In the special case where f is the constant function 1,

$$\int_C ds = \text{ arclength of } C.$$

- For scalar line integrals, the orientation (i.e. direction) of C does not matter.
- If C is made up of different pieces, you can compute the line integral over each piece separately and then add up the results.

Vector Line Integral

- Arises when integrating a *vector field* $\mathbf{F}$ over a curve C.
- Notationally, is denoted by something like

$$\int_C \mathbf{F}\cdot d\mathbf{s},\ \int_C (P\,dx + Q\,dy),\ \text{or}\ \int_C (P\,dx + Q\,dy + R\,dz)$$

  where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ in the 2-dimensional case or $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ in the 3-dimensional case.

- To compute directly, find parametric equations $\mathbf{x}(t), a \le t \le b$ for C and use

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

- Changing the orientation of C changes the sign of a vector line integral.
- If C is made up of different pieces, you can compute the line integral over each piece separately and then add up the results.

Green's Theorem

- Most importantly, only applies when $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a 2-dimensional vector field, meaning there is no $\mathbf{k}$ component and no dependence on z, and C is a curve in the xy-plane.
- Technically, only applies when C is a closed curve, $\mathbf{F}$ is continuous everywhere on the region D which C encloses, and C is oriented so that D is on its left. (But see below for ways around this.)
- Gives a way to compute a vector line integral by turning it into a double integral:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

- If C is oriented the wrong way, Green's Theorem still applies if we change the sign of the double integral.
- If C is not closed, Green's Theorem might still be applicable by "closing off the curve" C. Check the *Notes on Green's Theorem* handout for an explanation of this method.
- If F is not continuous everywhere on the region enclosed by C, Green's Theorem might still be applicable by "replacing the curve" C. Check the *Notes on Green's Theorem* handout for an explanation of this method.

Fundamental Theorem of Line Integrals

- Only applies if $\mathbf{F}$ is conservative, meaning $\mathbf{F} = \nabla f$ for some function f.
- Gives a way to compute a vector line integral using f:

$$\int_C \nabla f \cdot d\mathbf{s} = f(\text{end point of } C) - f(\text{start point of } C).$$

- In particular, if C is closed

$$\int_C \nabla f \cdot d\, s = 0.$$

- If C_1 and C_2 have the same start and endpoints,

$$\int_{C_1} \nabla f \cdot d\, s = \int_{C_2} \nabla f \cdot d\, s.$$

- To test if $\mathbf{F}$ is conservative, we can:

  1. Try to find the potential function f which will make $\mathbf{F} = \nabla f$. For instance, in the 2-dimensional case, if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ we set up

  $$f_x = P \text{ and } f_y = Q,$$

     and try to solve for the function f which will have these partial derivatives by the technique of taking "anti-partial derivatives". The same idea can be used in the 3-dimensional case.

  2. If $\operatorname{curl} \mathbf{F} \neq \mathbf{0}$, then $\mathbf{F}$ is definitely **not** conservative. In the 2-dimensional case where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ with no z dependence, this means that

  $$\text{if } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \neq 0, \text{ then } \mathbf{F} \text{ is definitely not conservative.}$$

  3. If $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and $\mathbf{F}$ is continuous on a simply-connected region (meaning a region with no holes), then $\mathbf{F}$ is conservative on that region. In the 2-dimensional case, this means that

  $$\text{if } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \text{ and } \mathbf{F} \text{ is continuous on a simply-connected region,}$$

     then $\mathbf{F}$ is conservative on that region.