Math 291-1: Final Exam
Northwestern University, Fall 2015

Name: ______________________________

1. (15 points) Determine, with justification, whether each of the following is true or false.
(a) If A and B are 2×2 matrices in reduced row-echelon form with the same image, then $A = B$.
(b) There exists a 2×2 matrix B such that $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) B = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$.
(c) Any 5-dimensional real vector space has a 3-dimensional subspace.

Problem	Score
1	
2	
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Total	

2. (10 points) Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Show that $\operatorname{rank}(A) = \operatorname{rank}(A \mid \mathbf{b})$ if and only if $\mathbf{b} \in \operatorname{im} A$, where $(A \mid \mathbf{b})$ denotes an augmented matrix.

3. (10 points) Show that if A and B are $n \times n$ matrices such that $AB = I$, then A and B are invertible. Hint: First show that B is invertible using some portion of the Amazingly Awesome Theorem.

4. (10 points) Let $p_1(x), p_2(x), p_3(x)$ be the polynomials

$$p_1(x) = 1 - x^2, \;\; p_2(x) = 2 + x, \;\; p_3(x) = 8 + 3x - 2x^2.$$

Show that $q(x) = a + bx + cx^2$ is in span $\{p_1(x), p_2(x), p_3(x)\}$ if and only if $a - 2b + c = 0$, and determine the dimension of this span.

5. (10 points) Let W be an affine subspace of $\mathbb{R}^2$ and let $\mathbf{b} \in W$. Show that

$$U = \{\mathbf{w} - \mathbf{b} \mid \mathbf{w} \in W\}$$

is a linear subspace of $\mathbb{R}^2$. You cannot just simply quote the homework problem which says this is true—you must work it out in this special case.

6. (10 points) Suppose V is a vector space over $\mathbb{K}$ and that $T : V \to V$ is a linear transformation. If $v \in V$ has the property that $T^2(v) \neq 0$ but $T^3(v) = 0$, show that $v, T(v), T^2(v)$ are linearly independent.

7. (10 points) Consider the function $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$T(A) = A + A^* \text{ for any } A \in M_2(\mathbb{C}),$$

where A^* denotes the conjugate trancspose of A. This is a linear transformation over $\mathbb{R}$, meaning a linear transformation when considering $M_2(\mathbb{C})$ as a **real** vector space. Determine the dimension of the image of T.