Math 291-1: Final Exam
Northwestern University, Fall 2016

Name: ______________________________

1. (15 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If A is a 2×2 matrix such that $A^2 = 0$, then $A = 0$.

(b) If A is a 3×3 matrix whose image is a plane, then A is not invertible.

(c) Any 5 elements of $P_3(\mathbb{R})$ are linearly dependent.

Problem	Score
1	
2	
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7	
Total	

2. (10 points) Let A be an $m \times n$ matrix. Show that the columns of A are linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

3. (10 points) Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are nonzero linearly independent vectors with the same length and that A is a 2×2 matrix satisfying

$$A\mathbf{v} = \mathbf{w} \text{ and } A\mathbf{w} = \mathbf{v}.$$

Show that A geometrically describes reflection across a line through the origin. Hint: First determine about which line this reflection would have to occur, and then show why A must have the effect of reflecting *any* vector across this line.

4. (10 points) Suppose A is an $n \times n$ matrix such that there exists a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$ satisfying

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1,\ A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2,\ \ldots,\ A\mathbf{v}_n = \lambda_n \mathbf{v}_n$$

for some *nonzero* scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Show that A is invertible.

5. (10 points) Let V be a vector space over $\mathbb{K}$. If $v_1, \ldots, v_k \in V$ and $w \in \operatorname{span}(v_1, \ldots, v_k)$, complete the following proof that

$$\operatorname{span}(v_1, \ldots, v_k, w) = \operatorname{span}(v_1, \ldots, v_k).$$

(Note: this requires showing that each side is a subset of the other, so if v is in the left side then it is also in the right side, and vice-versa.)

Proof. First let $b \in \operatorname{span}(v_1, \ldots, v_k, w)$. Then there exists $a_1, \ldots, a_n, a \in \mathbb{K}$ such that

$$b = \underline{\hspace{4cm}} + aw.$$

Since $w \in \operatorname{span}(v_1, \ldots, v_k)$ we have

$$w = \underline{\hspace{4cm}} \text{ for some } c_1, \ldots, c_k \in \mathbb{K}.$$

Thus

$$\begin{aligned} b &= \underline{\hspace{4cm}} + aw \\ &= a_1 v_1 + \cdots + a_k v_k + \underline{\hspace{4.5cm}} \\ &= \underline{\hspace{2.3cm}} v_1 + \cdots + \underline{\hspace{2.3cm}} v_k, \end{aligned}$$

so $b \in \underline{\hspace{3.5cm}}$.

Conversely suppose $b \in \operatorname{span}(v_1, \ldots, v_k)$. Then

$$\underline{\hspace{10cm}}.$$

But this can be written as

$$\underline{\hspace{6cm}},$$

so $\underline{\hspace{5cm}}$. We conclude that $\operatorname{span}(v_1, \ldots, v_k, w) = \operatorname{span}(v_1, \ldots, v_k)$ as claimed. $\square$

6. (10 points) Suppose V is a 2-dimensional vector space over $\mathbb{K}$ and that $T : V \to V$ is a linear transformation such that $T^3 = 0$. Show that $T^2 = 0$. Hint: if $v \in V$ is a vector such that $T^2 v \neq 0$, show first that $v, Tv, T^2 v$ are linearly independent.

7. (10 points) On the next page is a proof that the dimension of the subspace of $P_3(\mathbb{R})$ consisting of polynomials satisfying $p'(-1) = 0$ is 3 using the rank-nullity theorem. Using this as a guide, do the following. Let $W = \{p(x) \in P_4(\mathbb{R}) \mid p(1) = 0,\ p''(2) = p(1),\ \text{and } p'(3) = 0\}$.

(a) Find a linear transformation $T : P_4(\mathbb{R}) \to \mathbb{R}^3$ such that $W = \ker T$.

(b) Find the dimension of W.

Proof for Problem 10. The dimension of the subspace of $P_3(\mathbb{R})$ consisting of polynomials satisfying $p'(-1) = 0$ is 3.

Proof. Define $T : P_3(\mathbb{R}) \to \mathbb{R}$ by

$$T(p(x)) = p'(-1).$$

To be clear, T sends a polynomial to the value of its derivative at -1. This is a linear transformation since it is the composition of the transformation which takes the derivative of a polynomial with the transformation which evaluates a polynomial at -1, both of which are linear. Note that the kernel of T is precisely the subspace in question.

Since $T(x) = 1$, we have $1 \in \operatorname{im} T$ so $\operatorname{im} T$ is at least 1-dimensional. But since $\operatorname{im} T$ is contained in $\mathbb{R}$, we must thus have that $\operatorname{im} T = \mathbb{R}$. Hence $\dim \operatorname{im} T = 1$, so by the rank-nullity theorem we get

$$\dim \ker T = \dim P_3(\mathbb{R}) - \dim \operatorname{im} T = 4 - 1 = 3$$

as claimed. $\square$