Math 291-1: Final Exam
Northwestern University, Fall 2017

Name: ______________________________

1. (15 points) Determine whether each of the following statements is true or false, and provide justification for your answer.

(a) There is a 3×4 matrix whose columns are linearly independent.

(b) The complex vector space $M_3(\mathbb{C})$ has a 6-dimensional real subspace.

(c) The function $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(p(x)) = (p'(x))^2$ is a linear transformation.

Problem	Score
1	
2	
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7	
Total	

2. (10 points) Consider the linear system with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 2 & -2 & 3 & -5 & -1 \\ -3 & 3 & -6 & 12 & 3 \end{array}\right].$$

Find two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^4$ with the property that any solution of the system above can be written as

$$\begin{bmatrix} -3 \\ -2 \\ 2 \\ 1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some $c_1, c_2 \in \mathbb{R}$. Justify the reason why you're claimed vectors work.

3. (10 points) Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$ form a basis of $\mathbb{R}^4$ and that A is a 4×4 matrix with the property that

$$A\mathbf{v}_1 = \mathbf{v}_1,\ A\mathbf{v}_2 = \mathbf{v}_1,\ A\mathbf{v}_3 = \mathbf{v}_2,\ A\mathbf{v}_4 = \mathbf{v}_3.$$

Show that the image of A^4 is the entire span of $\mathbf{v}_1$.

4. (10 points) Suppose $A, B \in M_n(\mathbb{R})$. If AB is invertible, show that A and B are each invertible.

5. (10 points) Suppose U and W are subspaces of a vector space V over $\mathbb{K}$ which have only the zero vector in common. If $u_1, \ldots, u_k \in U$ are linearly independent and $w_1, \ldots, w_\ell \in W$ are linearly independent, show that $u_1, \ldots, u_k, w_1, \ldots, w_\ell$ are linearly independent. (This is not true if U and W have more than the zero vector in common, so you will definitely have to make use of this fact.)

6. (10 points) The *trace* $\operatorname{tr} A$ of a square matrix A is the sum of its diagonal entries:

$$\operatorname{tr}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} := a_{11} + a_{22} + \cdots + a_{nn}.$$

Find a basis for the subspace W of $M_4(\mathbb{R})$ consisting of symmetric matrices of trace zero:

$$W := \{A \in M_4(\mathbb{R}) \mid A^T = A \text{ and } \operatorname{tr} A = 0\}.$$

Don't forget to justify the fact that your claimed basis is actually a basis.

7. (10 points) Let U be the subspace of $P_4(\mathbb{R})$ consisting of all polynomials $p(x) \in P_4(\mathbb{R})$ satisfying both of the conditions

$$p''(2) = p(1) - p(2) \text{ and } p(5) = 0.$$

Determine, with justification, the dimension of U.