# Math 291-1: Final Exam Solutions
## Northwestern University, Fall 2015

**1.** Determine, with justification, whether each of the following is true or false.

(a) If $A$ and $B$ are $2\times 2$ matrices in reduced row-echelon form with the same image, then $A = B$.

(b) There exists a $2 \times 2$ matrix $B$ such that $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) B = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$.

(c) Any 5-dimensional real vector space has a 3-dimensional subspace.

*Solution.* (a) This is false. For instance,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

are both in reduced row-echelon form and have image equal to the $x$-axis, but are not equal.

(b) This is false. The matrix $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$ is invertible, and if the product of two square matrices is invertible then each matrix must be invertible as well, but $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is not invertible.

(c) This is true. If $V$ is 5-dimensional and $v_1, v_2, v_3, v_4, v_5$ is a basis of $V$, then $\operatorname{span}\{v_1, v_2, v_3\}$ will be a 3-dimensional subspace of $V$. ☐

**2.** Let $A$ be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Show that $\operatorname{rank}(A) = \operatorname{rank}(A \mid \mathbf{b})$ if and only if $\mathbf{b} \in \operatorname{im} A$, where $(A \mid \mathbf{b})$ denotes an augmented matrix.

*Proof.* The equality $\operatorname{rank}(A) = \operatorname{rank}(A \mid \mathbf{b})$ is true if and only if the final column of the reduced row-echelon form of $(A \mid \mathbf{b})$ does not have a pivot since all pivots in this reduced form would already be accounted for in the portion corresponding to $A$. This is true if and only if $\operatorname{rref}(A \mid \mathbf{b})$ does not contain a row of the form

$$\begin{pmatrix} 0 & \cdots & 0 & | & 1 \end{pmatrix},$$

which is true if and only if $A\mathbf{x} = \mathbf{b}$ has a solution, which is just another way of saying that $\mathbf{b} \operatorname{im} A$. ☐

**3.** Show that if $A$ and $B$ are $n \times n$ matrices such that $AB = I$, then $A$ and $B$ are invertible. Hint: First show that $B$ is invertible using some portion of the Amazingly Awesome Theorem.

*Proof 1.* Suppose $\mathbf{x} \in \mathbb{R}^n$ satisfies $B\mathbf{x} = \mathbf{0}$. Then $AB\mathbf{x} = \mathbf{0}$ so $\mathbf{x} = \mathbf{0}$ since $AB = I$. Thus the only solution of $B\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, so $B$ is invertible. Hence $B^{-1}$ exists, so $AB = I$ gives

$$ABB^{-1} = IB^{-1} \text{ and hence } A = B^{-1}.$$

Since the inverse of an invertible matrix is invertible, $A$ is invertible as well. ☐

*Proof 2.* We know $\operatorname{rank}(AB) \leq \operatorname{rank} B$ (as shown on the practice problems) and $\operatorname{rank}(AB) \leq \operatorname{rank} A$ (as shown on the homework). Thus since $AB = I$, $\operatorname{rank}(AB) = n$ so $n \leq \operatorname{rank} B$ and $n \operatorname{rank} A$. But each of these ranks is also at most $n$, so $\operatorname{rank} B = n$ and $\operatorname{rank} A = n$, which means that both $A$ and $B$ are invertible. ☐

**4.** Let $p_1(x), p_2(x), p_3(x)$ be the polynomials

$$p_1(x) = 1 - x^2,\ \ p_2(x) = 2 + x,\ \ p_3(x) = 8 + 3x - 2x^2.$$

Show that $q(x) = a + bx + cx^2$ is in $\operatorname{span}\{p_1(x), p_2(x), p_3(x)\}$ if and only if $a - 2b + c = 0$, and determine the dimension of this span.

*Proof.* We have that $q(x)$ is in the span of the given polynomials if and only if there exists scalars $c_1, c_2, c_3$ such that

$$c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = a + bx + cx^2,$$

which after plugging in for $p_1, p_2, p_3$ and regrouping gives

$$(c_1 + 2c_2 + 8c_3) + (c_2 + 3c_3)x + (-c_1 - 2c_3)x^2 = a + bx + cx^2.$$

Thus $a, b, c$ must have the property that the linear system

$$\begin{aligned} c_1 + 2c_2 + 8c_3 &= a \\ c_2 + 3c_3 &= b \\ -c_1 \qquad - 2c_3 &= c \end{aligned}$$

has a solution. Row operations give:

$$\begin{pmatrix} 1 & 2 & 8 & \mid & a \\ 0 & 1 & 3 & \mid & b \\ -1 & 0 & -2 & \mid & c \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 8 & \mid & a \\ 0 & 1 & 3 & \mid & b \\ 0 & 0 & 0 & \mid & a+c-2b \end{pmatrix},$$

so the given system has a solution if and only if $a + c - 2b = 0$ as claimed.

Under the isomorphism $P_2(\mathbb{R}) \to \mathbb{R}^3$ given by

$$a + bx + cx^2 \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

the span of the given polynomials corresponds to the span of

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 3 \\ -2 \end{pmatrix}.$$

Based on the row operations above, this span of these three vectors is 2-dimensional since the reduced echelon form will have 2 pivots, so span $\{p_1(x), p_2(x), p_3(x)\}$ is 2-dimensional as well. $\square$

**5.** Let $W$ be an affine subspace of $\mathbb{R}^2$ and let $\mathbf{b} \in W$. Show that

$$U = \{\mathbf{w} - \mathbf{b} \mid \mathbf{w} \in W\}$$

is a linear subspace of $\mathbb{R}^2$. You cannot just simply quote the homework problem which says this is true—you must work it out in this special case.

*Proof 1.* This proof just repeats the proof of the more general situation given in the homework. First, since $\mathbf{b} \in W$ we have

$$\mathbf{b} - \mathbf{b} = \mathbf{0} \in U,$$

so $U$ contains the zero vector. If $\mathbf{w}_1 - \mathbf{b}, \mathbf{w}_1 - \mathbf{b} \in U$ with $\mathbf{w}_1, \mathbf{w}_2 \in W$, then

$$(\mathbf{w}_1 - \mathbf{b}) + (\mathbf{w}_1 - b) = (\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{b}) - \mathbf{b} \in U$$

since $\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{b} \in W$ given that $W$ is closed under affine combinations; this shows that $U$ is closed under addition. Finally, if $\mathbf{w} - \mathbf{b} \in W$ and $a \in \mathbb{R}$, then

$$a(\mathbf{w} - \mathbf{b}) = a\mathbf{w} - a\mathbf{b} = a\mathbf{w} + (1-a)\mathbf{b} - \mathbf{b} \in U$$

since $a\mathbf{w} + (1-a)\mathbf{b} \in W$ given that $W$ is closed under affine combinations. Hence $U$ is closed under scalar multiplication, so it is a subspace of $\mathbb{R}^2$. $\square$

*Proof 2.* In the case of $\mathbb{R}^2$ there is another proof we can give which since we know what all affine subspaces of $\mathbb{R}^2$ must look like. If $W = \{\mathbf{b}\}$ consists of a single point, then the only thing in $U$ is $\mathbf{b} - \mathbf{b} = \mathbf{0}$, and $U = \{\mathbf{0}\}$ is a linear subspace of $\mathbb{R}^2$. If $W = \mathbb{R}^2$, then $U$ is still all of $\mathbb{R}^2$ since subtracting $\mathbf{b}$ from points of $\mathbb{R}^2$ still results in all possible points of $\mathbb{R}^2$. Hence in this case $U$ is also a linear subspace of $\mathbb{R}^2$.

Finally, if $W$ is a line $ax + by = c$, passing through the origin or not, then $\mathbf{b} = \left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right)$ on this satisfies $ax_0 + by_0 = c$. Thus if $\mathbf{w} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is any other point on this line,

$$\mathbf{w} - \mathbf{b} = \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

which satisfies $a(x - x_0) + b(y - y_0) = 0$. Thus $U$ in this case is the line $ax + by = 0$, which *does* pass through the origin and is thus a linear subspace of $\mathbb{R}^2$. $\square$

**6.** Suppose $V$ is a vector space over $\mathbb{K}$ and that $T : V \to V$ is a linear transformation. If $v \in V$ has the property that $T^2(v) \neq 0$ but $T^3(v) = 0$, show that $v, T(v), T^2(v)$ are linearly independent.

*Proof.* Suppose $c_1, c_2, c_3 \in \mathbb{K}$ satisfy

$$c_1 v + c_2 T(v) + c_3 T^2(v) = 0.$$

Applying $T$ to both sides gives

$$c_1 T(v) + c_2 T^2(v) + c_3 T^3(v) = T(0),$$

which becomes

$$c_1 T(v) + c_2 T^2(v) = 0$$

since $T^3(v) = 0$ and $T(0) = 0$. Applying $T$ to this new equation gives

$$c_1 T^2(v) = 0.$$

Since $T^2(v) \neq 0$, this means that $c_1 = 0$, which then turns the previous equation into

$$c_2 T^2(v) = 0.$$

Again since $T^2(v) \neq 0$, this means $c_2 = 0$, which turns the original equation into

$$c_3 T^2(v) = 0,$$

which gives $c_3 = 0$. Thus $c_1 v + c_2 T(v) + c_3 T^2(v) = 0$ implies $c_1 = c_2 = c_3 = 0$, so $v, T(v), T^2(v)$ are linearly independent. $\square$

**Remark.** Here is a nice use of this fact. The claim is that if $A$ is a $2 \times 2$ matrix for which $A^3 = 0$, then $A^2 = 0$. Indeed, if $A^2 \neq 0$, there is a vector $v$ such that $A^2 v \neq 0$. Since $A^3 = 0$, so $A^3 v = 0$ and hence the result of this problem shows that $v, Av, A^2 v$ are linearly independent, which is nonsense since these are vectors in $\mathbb{K}^2$ and you can't have more linearly independent vectors than the dimension of your space. A generalization of this problem can be used to show that if $A$ is $n \times n$ and satisfies $A^k = 0$ for some $k$, then in fact $A^n = 0$.

**7.** Consider the function $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$T(A) = A + A^* \text{ for any } A \in M_2(\mathbb{C}),$$

where $A^*$ denotes the conjugate trancspose of $A$. This is a linear transformation over $\mathbb{R}$, meaning a linear transformation when considering $M_2(\mathbb{C})$ as a **real** vector space. Determine the dimension of the image of $T$.

*Proof 1.* Here is a rank-nullity approach. Set

$$A = \begin{pmatrix} a+ib & c+id \\ e+if & g+ih \end{pmatrix}.$$

Then $A \in \ker T$ if and only if $A^* = -A$ (such an $A$ is called *skew-Hermitian*), which requires

$$\begin{pmatrix} a-ib & e-if \\ c-id & g-ih \end{pmatrix} = \begin{pmatrix} -a-ib & -c-id \\ -e-if & -g-ih \end{pmatrix}.$$

Comparing coefficients gives $a = 0, g = 0, e = -c, f = d$, so a matrix in the kernel is of the form

$$\begin{pmatrix} ib & c+id \\ -c+id & ih \end{pmatrix} = b\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + h\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}.$$

The four matrices on the right form a basis for $\ker T$, so $\dim(\ker T) = 4$. Since $\dim M_2(\mathbb{C}) = 8$ as a vector space over $\mathbb{R}$, rank-nullity gives $\dim(\operatorname{im} T) = 4$. □

*Proof 2.* We can also determine a basis for the image explicitly. (The image is actually the space of all $2 \times 2$ Hermitian matrices, which is a consequence of this problem.)

Using the same notation as before, an element of the image looks like

$$A + A^* = \begin{pmatrix} a+ib & c+id \\ e+if & g+ih \end{pmatrix} + \begin{pmatrix} a-ib & e-if \\ c-id & g-ih \end{pmatrix} = \begin{pmatrix} 2a & (c+e)+i(d-f) \\ (e+c)+i(f-d) & 2g \end{pmatrix}.$$

Breaking this up gives

$$\begin{pmatrix} 2a & (c+e)+i(d-f) \\ (e+c)+i(f-d) & 2g \end{pmatrix} = 2a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (c+e)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (d-f)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + 2g\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so the four matrices on the right give a basis for $\operatorname{im} T$, which is thus 4-dimensional. (Note that these four matrices also span the space of $2 \times 2$ Hermitian matrices as we saw on a homework, so the image is the space of Hermitian matrices.) □