Math 291-3: Final Exam Solutions
Northwestern University, Spring 2017

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If $f(x, y)$ is continuous everywhere except on the set of points satisfying $x^2 + 2y^2 \leq 1$, then f is integrable over the rectangle $[-3, 3] \times [-3, 3]$.

(b) If $\mathbf{F}$ is C^1 on an open set $U \subseteq \mathbb{R}^2$ and $\operatorname{curl} \mathbf{F} = \mathbf{0}$ on U, then $\mathbf{F}$ is conservative on U.

(c) If S_1 and S_2 are oriented surfaces with the same boundary and which induce the same orientation on that boundary, then $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ for any C^1 vector field $\mathbf{F}$.

Solution. (a) This is false. The set of points satisfying $x^2 + 2y^2 \leq 1$ is the region enclosed by an ellipse, so it has positive area and hence is not of measure zero.

(b) This is false. The field $\mathbf{F} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2+y^2}$ on U the xy-plane with the origin removed has curl zero but is not conservative on U.

(c) This is true. By Stokes' Theorem, both surface integrals equal $\int_{\partial S_1 = \partial S_2} \mathbf{F} \cdot d\mathbf{s}$. $\square$

2. Consider the following iterated integral:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{-1}^{-\sqrt{x^2+y^2}} y^2 \, dz \, dx \, dy.$$

(a) Rewrite this as an iterated integral in cylindrical coordinates.

(b) Rewrite this as an iterated integral in spherical coordinates.

Solution. The region of integration is the portion of the solid lying below the downward-opening cone $z = -\sqrt{x^2+y^2}$ and above the plane $z = -1$ which lies below the first quadrant of the xy-plane. (So, where x and y are both nonnegative.)

(a) In cylindrical coordinates we have

$$\int_0^{\pi/2} \int_0^1 \int_{-1}^{-r} y^2 r \, dz \, dr \, d\theta,$$

where we use the fact that the cone $z = -\sqrt{x^2+y^2}$ becomes $z = -r$ in cylindrical coordinates. We could also write this with respect to the order $dr\,dz\,d\theta$, in which case we get

$$\int_0^{\pi/2} \int_{-1}^0 \int_0^{-z} y^2 r \, dr \, dz \, d\theta.$$

(b) In spherical coordinates the cone is given by $\phi = 3\pi/4$, so we get

$$\int_0^{\pi/2} \int_{3\pi/4}^{\pi} \int_0^{-1/\cos\phi} \rho^2 \sin^2\phi \sin^2\theta \phi^2 \sin\phi \, d\rho \, d\phi \, d\theta,$$

where we use the fact that the plane $z = -1$ becomes $\phi \cos\phi = -1$ in spherical coordinates. $\square$

3. Suppose A is an invertible $n \times n$ matrix. Let D be a compact region in $\mathbb{R}^n$ such that the constant function 1 is integrable over D and let $A(D)$ denote the image of D under the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ defined by A. Show that

$$\operatorname{Vol} A(D) = |\det A| \operatorname{Vol}(D).$$

(This is the geometric interpretation of the determinant as an expansion factor we first introduced last quarter, but the point is to justify this using material from this quarter.)

Proof. We have

$$\operatorname{Vol} A(D) = \int_{A(D)} d\mathbf{x}.$$

By the change of variables formula, this equals

$$\int_D |\det DA|\, d\mathbf{x}.$$

The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ has Jacobian matrix A, so the integral above becomes

$$\int_D |\det A|\, d\mathbf{x} = |\det A| \int_D d\mathbf{x} = |\det A| \operatorname{Vol} D.$$

Hence $\operatorname{Vol} A(D) = |\det A| \operatorname{Vol} D$ as claimed. ☐

4. Let C be the curve where the cylinder $y^2 + z^2 = 1$ and the plane $x = y$ intersect. Show that C is smooth everywhere. (Recall that a curve is said to be *smooth* at the points where its tangent vector is nonzero.)

Proof. We parametrize the given curve using

$$\mathbf{x}(t) = (\cos t, \cos t, \sin t),\ \ 0 \leq t \leq 2\pi,$$

which come from parametrizing the y, z directions from the cylinder and using the plane equation to find x. We have

$$\mathbf{x}'(t) = (-\sin t, -\sin t, \cos t).$$

There are no values of t for which both $\cos t$ and $-\sin t$ are simultaneously zero, so $\mathbf{x}'(t)$ is never zero and hence C is smooth. ☐

5. Show that a C^1 vector field $\mathbf{F}$ on $\mathbb{R}^n$ has path-independent line integrals if and only if its line integral over any closed smooth C^1 curve is zero. (Recall that $\mathbf{F}$ having path-independent line integrals means that if C_1 and C_2 are smooth C^1 curves which begin at the same point and end at the same point, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$. You may not use the fact that both of these properties are equivalent to $\mathbf{F}$ being conservative.)

Proof. This was the Warm-Up for Lecture 23 in the lecture notes, which then reference my Math 290-3 lecture notes for the actual proof. I'll direct you to find the details there. ☐

6. Compute the line integral

$$\int_C (x \sin e^x - xz)\, dx - 2xy\, dy + (z^2 + y)\, dz$$

where C is the curve consisting of the line segment from $(2, 0, 0)$ to $(0, 2, 0)$, followed by the line segment from $(0, 2, 0)$ to $(0, 0, 2)$, followed by the line segment from $(0, 0, 2)$ to $(2, 0, 0)$. Hint: C lies on the plane $x + y + z = 2$.

Solution. Let D be the region of the plane $x + y + z = 2$ enclosed by C, so that $C = \partial D$. Stokes' Theorem gives

$$\int_{\partial D} (x \sin e^x - xz)\, dx - 2xy\, dy + (z^2 + y)\, dz = \iint_D d[(x \sin e^x - xz)\, dx - 2xy\, dy + (z^2 + y)\, dz]$$

where we orient D with upward-pointing normal vectors. The exterior derivative on the right side is

$$dy \wedge dz - x\,dz \wedge dx - 2y\,dx \wedge dy.$$

Or, phrasing this all in terms of vector fields instead, the curl of the vector field

$$\mathbf{F} = (x \sin e^x - xz, -2xy, z^2 + y)$$

is

$$\operatorname{curl} \mathbf{F} = (1, -x, -2y).$$

Thus the required surface integral is

$$\iint_D (1, -x, -2y)\, d\mathbf{S}.$$

We parametrize D using

$$\mathbf{X}(x, y) = (x, y, 2 - x - y),\ (x, y) \in D'$$

where D' is the triangular region in the xy-plane bounded by the coordinate axes and the line $x + y = 2$. We get:

$$\mathbf{X}_x \times \mathbf{X}_y = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1),$$

so

$$\begin{aligned}\iint_D (1, -x, -2y)\, d\mathbf{S} &= \int_0^2 \int_0^{2-x} (1, -x, -2y) \cdot (1, 1, 1)\, dy\, dx \\ &= \int_0^2 \int_0^{2-x} (1 - x - 2y)\, dy\, dx \qquad = -2.\end{aligned}$$

Thus the given line integral has value -2. □

7. Compute the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where

$$\mathbf{F} = (3x - ye^{\cos z})\,\mathbf{i} + (e^{x^{10} z^8} - 2yz)\,\mathbf{j} + (z^2 + ye^x)\,\mathbf{k}$$

where S is the portion of the cylinder $x^2 + y^2 = 1$ which lies between $z = 0$ and $z = 1$, oriented with inward pointing normal vectors.

Solution. Let S_1 be the unit disk in the plane $z = 0$ and S_2 the unit disk in the plane $z = 1$. We give S_1 the upward orientation and S_2 the downward orientation. Then $S + S_1 + S_2$ is a closed surface with inward orientation enclosing a solid E, so Gauss's Theorem gives

$$\iint_{S+S_1+S_2} \mathbf{F} \cdot d\mathbf{S} = -\iiint_E \operatorname{div} \mathbf{F}\, dV = -\iiint_E (3 - 2z + 2z)\, dV = -3 \operatorname{Vol}(E) = -3\pi.$$

Now, the integral we want is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S+S_1+S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

The integral over S_1 is:

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{k}\, dS = \iint_{S_1} ye^x\, dS$$

where we use the fact that $z = 0$ on S_1 to say that $z^2 + ye^x = ye^x$. This integral is zero, since ye^x is odd with respect to y and S_1 is symmetric with respect to y.

The integral over S_2 is:

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot \mathbf{k} \, dS = \iint_{S_2} (1 + ye^x) \, dS$$

where we use the fact that $z = 1$ on S_2. The ye^x term integrates to zero for the same reason as before, and the constant 1 integrates to the area of S_2, which is π. Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -3\pi - 0 - \pi = -4\pi.$$

□