Math 291-2: Final Exam Solutions
Northwestern University, Winter 2017

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If a linear transformation preserves the angle between any two vectors, then it is orthogonal.

(b) If $\mathbf{v}$ is an eigenvector of a square matrix A, then $\mathbf{v}$ is also an eigenvector of A^2.

(c) The level curves of $f(x, y) = x^2 - y^2$ are all hyperbolas.

Solution. (a) This is false. For instance, $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ simply scales the length of any vector by a factor of 2, which doesn't alter angles, but is not orthogonal.

(b) This is true. Say that λ is the corresponding eigenvalue. Then $A\mathbf{x} = \lambda\mathbf{x}$, so $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$, so $\mathbf{x}$ is eigenvector of A^2 with eigenvalue λ^2.

(c) This is false. The level curve at $z = 0$ is $0 = x^2 - y^2$, which describes the pair of lines $y = \pm x$. ◻

2. Suppose A is an $n \times n$ matrix and that A^T is its transpose.

(a) Show that $A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y}$. Hint: Work out what this becomes when $\mathbf{x} = \mathbf{e}_i$ and $\mathbf{y} = \mathbf{e}_j$.

(b) Show that $(AB)^T = B^T A^T$ for any $n \times n$ matrix B.

Proof. (a) First we have:

$$A\mathbf{e}_i \cdot \mathbf{e}_j = (i\text{-th column of } A) \cdot \mathbf{e}_j = j\text{-th entry in } i\text{-th column of } A$$

and

$$\mathbf{e}_i \cdot A^T\mathbf{e}_j = \mathbf{e}_i \cdot (j\text{-th column of } A^T) = i\text{-th entry in } j\text{-th column of } A^T.$$

Since the j-th entry in the i-th column of A is the same as the i-th entry in the j-th column of A^T, we get that $A\mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{e}_i \cdot A^T\mathbf{e}_j$ for any i, j.

Now, take any $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$ and $\mathbf{y} = y_1\mathbf{e}_1 + \cdots + y_n\mathbf{e}_n$. Then

$$\begin{aligned}
A\mathbf{x} \cdot \mathbf{y} &= (x_1 A\mathbf{e}_1 + \cdots + x_n A\mathbf{e}_n) \cdot (y_1\mathbf{e}_1 + \cdots + y_n\mathbf{e}_n) \\
&= \sum_{i,j} x_i y_j A\mathbf{e}_i \cdot \mathbf{e}_j \\
&= \sum_{i,j} x_i y_i \mathbf{e}_i \cdot A^T\mathbf{e}_j \\
&= (x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \cdot (y_1 A^T\mathbf{e}_1 + \cdots + y_n A^T\mathbf{e}_n) \\
&= \mathbf{x} \cdot A^T\mathbf{y}
\end{aligned}$$

as claimed

(b) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$(AB)\mathbf{x} \cdot \mathbf{y} = A(B\mathbf{x}) \cdot \mathbf{y} = B\mathbf{x} \cdot A^T\mathbf{y} = \mathbf{x} \cdot B^T(A^T\mathbf{y}) = \mathbf{x} \cdot (B^T A^T)\mathbf{y}.$$

Thus $B^T A^T$ is the matrix satisfying the property

$$(AB)\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot (AB)^T\mathbf{y}$$

required of $(AB)^T$, so $(AB)^T = B^T A^T$ as desired. ◻

3. In this problem you can use whichever definition of the determinant you like, but make clear which definition you are using.

(a) Show that a matrix with two identical rows has determinant zero.

(b) Show that the row operation which replaces row $\mathbf{r}_j$ of a matrix by $c\mathbf{r}_i + \mathbf{r}_j$ (where c is a scalar and $\mathbf{r}_i$ is another row) does not change the value of the determinant of that matrix.

Proof. (a) Using the characterization of the determinant as the unique multilinear, alternating map $\det : M_n(\mathbb{R}) \to \mathbb{R}$ which sends I to 1, we have that

$$\det A = -\det(A \text{ with the two identical rows swapped}) = -\det A$$

by the alternating property. Thus $2\det A = 0$, so $\det A = 0$.

(b) Consider A^T. By multilinearity we have:

$$\det \begin{bmatrix} \cdots & c\mathbf{r}_i^T + \mathbf{r}_j^T & \cdots \end{bmatrix} = c \det \begin{bmatrix} \cdots & \mathbf{r}_i^T & \cdots \end{bmatrix} + \det \begin{bmatrix} \cdots & \mathbf{r}_j^T & \cdots \end{bmatrix}.$$

But the first matrix on the right has repeated columns (the i- and j-th columns are both $\mathbf{r}_i^T$), so applying part (a) to the transpose says that this determinant is zero. Thus

$$\det \begin{bmatrix} \cdots & c\mathbf{r}_i^T + \mathbf{r}_j^T & \cdots \end{bmatrix} = \det \begin{bmatrix} \cdots & \mathbf{r}_j^T & \cdots \end{bmatrix},$$

and taking transposes (which does not affect the value of the determinant) gives the required claim. $\square$

4. Suppose A is a symmetric 3×3 matrix with eigenvalues $1, 1, -3$ and associated eigenvectors

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \text{ respectively.}$$

(a) Verify that the orthonormal eigenvectors obtained by applying the Gram-Schmidt process to these vectors are:

$$\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \text{ and } \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}.$$

(b) Compute $A^2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution. (a) Call these vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Then Gram-Schmidt gives

$$\begin{aligned}
\mathbf{b}_1 &= \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\
\mathbf{b}_2 &= \mathbf{v}_2 - \operatorname{proj}_{\mathbf{b}_1} \mathbf{v}_2 \\
&= \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \\
\mathbf{b}_3 &= \mathbf{v}_3 - \operatorname{proj}_{\mathbf{b}_1} \mathbf{v}_3 - \operatorname{proj}_{\mathbf{b}_2} \mathbf{v}_3 \\
&= \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} - \frac{0}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \frac{0}{9} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.
\end{aligned}$$

Dividing each of these by their lengths then gives the claimed vectors. Note that the third step was unnecessary since we know $\mathbf{v}_3$ is already orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$ since it corresponds to a different eigenvalue.

(b) We first express $\mathbf{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ as a linear combination of the resulting orthonormal eigenvectors $\mathbf{u}_1, \mathbf{u}_2\mathbf{u}_3$ from (a):

$$\mathbf{x} = (\mathbf{x}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x}\cdot\mathbf{u}_2)\mathbf{u}_2 + (\mathbf{x}\cdot\mathbf{u}_3)\mathbf{u}_3 = \frac{5}{3}\begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1/3\\2/3\\-2/3\end{bmatrix}.$$

Using the fact that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with eigenvalues $1, 1, -3$ respectively, they are also eigenvalues of A^2 with eigenvalues $1, 1, 9$ respectively, so:

$$\begin{aligned}
A^2\begin{bmatrix}1\\1\\1\end{bmatrix} &= \frac{5}{3}A^2\begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix} + \frac{1}{3}A^2\begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix} + \frac{1}{3}A^2\begin{bmatrix}1/3\\2/3\\-2/3\end{bmatrix}\\
&= \frac{5}{3}\begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix} + \frac{9}{3}\begin{bmatrix}1/3\\2/3\\-2/3\end{bmatrix}\\
&= \begin{bmatrix}17/9\\25/9\\-7/9\end{bmatrix}.
\end{aligned}$$

$\square$

5. Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable and has the property that $Df(\mathbf{x})$ is the same matrix A for every $\mathbf{x}$. Show that f is an affine transformation, i.e. has the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some $\mathbf{b}$. Hint: What is the Jacobian matrix of the function $g(\mathbf{x}) = f(\mathbf{x}) - A\mathbf{x}$ at any $\mathbf{x}$?

Proof. Since the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ has Jacobian matrix A at any point, we get that the Jacobian matrix of $g(\mathbf{x}) = f(\mathbf{x}) - A\mathbf{x}$ at any $\mathbf{x}$ is

$$Dg(\mathbf{x}) = Df(\mathbf{x}) - A = A - A = 0.$$

Thus g has Jacobian matrix 0 everywhere, which implies that g is constant: there exists $\mathbf{b} \in \mathbb{R}^n$ such that $g(\mathbf{x}) = \mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^m$. Hence $f(\mathbf{x}) - A\mathbf{x} = \mathbf{b}$ for all $\mathbf{x}$, so $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ is affine as required. $\square$

6. Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are each differentiable and let fg be the function defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$. Complete the following proof of the product rule:

$$D(fg)(\mathbf{x}) = g(\mathbf{x})Df(\mathbf{x}) + f(\mathbf{x})Dg(\mathbf{x}).$$

Proof. Let $h : \mathbb{R}^n \to$ __ be the function defined by

$$h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

and $m : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$m(x, y) = \_\_.$$

Then $(fg)(\mathbf{x})$ is the composition $\underline{\qquad\quad}$. By the chain rule we have

$$D(fg)(\mathbf{x}) = \underline{\qquad\qquad\qquad}.$$

We compute:

$$Dm(x,y) = \begin{bmatrix} y & x \end{bmatrix} \text{ and } Dh(\mathbf{x}) = \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix},$$

so

$$D(fg)(\mathbf{x}) = \begin{bmatrix} \underline{\qquad} & \underline{\qquad} \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} = \underline{\qquad\qquad\qquad\qquad\qquad}$$

for any $\mathbf{x} \in \mathbb{R}^n$ as claimed. $\square$

Proof. Let $h : \mathbb{R}^n \to \mathbb{R}^2$ be the function defined by

$$h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

and $m : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$m(x,y) = xy.$$

Then $(fg)(\mathbf{x})$ is the composition $m(h(\mathbf{x}))$. By the chain rule we have

$$D(fg)(\mathbf{x}) = Dm(h(\mathbf{x}))Dh(\mathbf{x}).$$

We compute:

$$Dm(x,y) = \begin{bmatrix} y & x \end{bmatrix} \text{ and } Dh(\mathbf{x}) = \begin{bmatrix} Df(\mathbf{x}) \\ Dg(\mathbf{x}) \end{bmatrix},$$

so

$$D(fg)(\mathbf{x}) = \begin{bmatrix} g(\mathbf{x}) & f(\mathbf{x}) \end{bmatrix} \begin{bmatrix} Df(\mathbf{x}) \\ Dg(\mathbf{x}) \end{bmatrix} = g(\mathbf{x})Df(\mathbf{x}) + f(\mathbf{x})Dg(\mathbf{x})$$

for any $\mathbf{x} \in \mathbb{R}^n$ as claimed. $\square$

7. Let $f(x,y) = xe^{x-y} + 2x^2y$. Suppose that at the point $(1,1)$ the steepest part of the graph of f has slope M. Find the directions (in terms of explicit vectors) in which the directional derivative of f at $(1,1)$ is $\frac{M}{\sqrt{2}}$.

Solution. Since f is differentiable at $(1,1)$, we have

$$D_{\mathbf{u}}f(1,1) = \nabla f(1,1) \cdot \mathbf{u}$$

for any direction vector $\mathbf{u}$. Since $M = \|\nabla f(1,1)\|$, we want the direction for which

$$\nabla f(1,1) \cdot \mathbf{u} = \|\nabla f(1,1)\| \cos\theta = \|\nabla f(1,1)\| \frac{1}{\sqrt{2}}.$$

This occurs when $\cos\theta = \frac{1}{\sqrt{2}}$, so when $\theta = \pm\frac{\pi}{4}$. Thus $\mathbf{u}$ should be a vector making an angle of $\pm\frac{\pi}{4}$ with $\nabla f(1,1)$. We have

$$\nabla f(x,y) = \left\langle e^{x-y} + xe^{x-y} + 4xy, -xe^{x-y} + 2x^2 \right\rangle, \text{ so } \nabla f(1,1) = \langle 6, 1 \rangle.$$

Hence the required directions are

$$\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ 7/\sqrt{2} \end{bmatrix}$$

and

$$\begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/\sqrt{2} \\ -5/\sqrt{2} \end{bmatrix},$$

or the result of dividing these by their lengths if we want unit vectors as direction vectors. $\square$