Math 291-1: Midterm 1

Northwestern University, Fall 2015

Name: ____

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) A linear system of 3 equations with 2 variables cannot have infinitely many solutions.

(b) If A, B are 2×2 matrices such that the set solutions of $A\mathbf{x} = \mathbf{0}$ is spanned by $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and the set of solutions of $B\mathbf{x} = \mathbf{0}$ is spanned by $\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right)$, then A and B have the same reduced row-echelon form.

2. (10 points) Consider the linear system with augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 & -1 & | & 0 \\ 2 & -2 & 3 & -5 & | & -1 \\ -3 & 3 & -6 & 12 & | & 3 \end{pmatrix}.$$

The vector

$$\mathbf{x}_0 = \begin{pmatrix} -3 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

gives a solution. Find two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^4$ with the property that any solution of the system above can be written as

$$\mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some $c_1, c_2 \in \mathbb{R}$. Justify the reason why you're claimed vectors work.

3. (10 points) Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly independent but that for some $\mathbf{w} \in \mathbb{R}^n$, the vectors

$$\mathbf{v}_1 + \mathbf{w}, \ldots, \mathbf{v}_k + \mathbf{w}$$

obtained by adding $\mathbf{w}$ to each $\mathbf{v}_i$ are linearly dependent. Show that $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

4. (10 points) Prove that

$$a(\mathbf{v}_1 + \cdots + \mathbf{v}_n) = a\mathbf{v}_1 + \cdots + a\mathbf{v}_n$$

for any *complex* scalar $a \in \mathbb{C}$ and $n \geq 2$ complex vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{C}^2$. You cannot take it for granted that multiplication of complex numbers is distributive; you must prove this if you need it.

5. (10 points) Suppose that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$ span $\mathbb{R}^4$. Let A be the 4×4 matrix having $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as columns. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$ are vectors such that $A\mathbf{x} = A\mathbf{y}$, show that $\mathbf{x} = \mathbf{y}$. Hint: Of which equation is $\mathbf{x} - \mathbf{y}$ a solution?