Math 291-1: Midterm 1
Northwestern University, Fall 2016

Name: ______________________________

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample. (A counterexample is a specific example in which the given claim is indeed false.)

(a) If $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{C}^2$ are complex vectors which are linearly dependent over $\mathbb{C}$, then $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are also linearly dependent over $\mathbb{R}$. (Recall that the distinction between "over $\mathbb{C}$" and "over $\mathbb{R}$" is whether or not we allow arbitrary complex scalars or only real scalars as coefficients.)

(b) If A, B are matrices for which $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solutions, then $A = B$.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$. Show that $\mathbf{b} \in \mathbb{R}^3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ if and only if $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_2$. (This shows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_2$ have the same span.)

Bonus (2 extra points): Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Show that $\mathbf{b} \in \mathbb{R}^n$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ if and only if $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \ldots, \mathbf{v}_k - \mathbf{v}_{k-1}$, where each vector in this new list except the first is of the form $\mathbf{v}_i - \mathbf{v}_{i-1}$ for $i = 1, \ldots, k$. Note that the original problem is a special case of the Bonus, so doing the Bonus alone will get you the full 12 points.

3. (10 points) Consider the system of linear equations in four variables with augmented matrix

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 3 & 3 \\ -3 & -9 & -4 & -5 & -7 \\ 2 & 6 & 4 & 6 & 6 \end{bmatrix}.$$

Show that any solution $\mathbf{x} \in \mathbb{R}^4$ of this system can be written as

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

for some $a, b \in \mathbb{R}$. Hint: You can use the result of the Bonus in Problem 2 without justification.

4. (10 points) Let A be a 2×2 matrix. Prove that

$$A(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n) = c_1A\mathbf{x}_1 + \cdots + c_nA\mathbf{x}_n$$

for any $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^2$ and any $c_1, \ldots, c_n \in \mathbb{R}$. You cannot take $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ nor $A(c\mathbf{x}) = cA\mathbf{x}$ for granted, and must justify these facts first if you need them.

5. (10 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}$ and let A be the matrix with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns. If there exists $\mathbf{b} \in \mathbb{R}^3$ for which $A\mathbf{x} = \mathbf{b}$ has no solution, show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.