Math 291-1: Midterm 1
Northwestern University, Fall 2017

Name: ______________________________

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample. (A counterexample is a specific example in which the given claim is indeed false.)

(a) If A and B are 2×2 matrices such that $\text{rref}(A) = \text{rref}(B)$ and $A\begin{bmatrix}\pi \\ e\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$, then $B\begin{bmatrix}2\pi \\ 2e\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$.

(b) If $\mathbf{w}, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^2$ and $\mathbf{w}$ is a complex linear combination of $\mathbf{z}_1, \mathbf{z}_2$, then $\mathbf{w}$ is also a real linear combination of $\mathbf{z}_1, \mathbf{z}_2$. (Recall that the distinction between complex and real linear combinations comes in the types of scalars we allow as coefficients.)

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^n$ and that $\mathbf{u} \in \mathbb{R}^n$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Show that $\mathbf{u}$ can also be written as a linear combination of

$$\mathbf{v}_1 - \mathbf{v}_3,\ \mathbf{v}_2,\ \mathbf{v}_2 - \mathbf{v}_4,\ \mathbf{v}_3 - \mathbf{v}_4.$$

3. (10 points) If $n \geq 2$, show that for any $a \in \mathbb{R}$ and any $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^2$, we have

$$a(\mathbf{v}_1 + \cdots + \mathbf{v}_n) = a\mathbf{v}_1 + \cdots + a\mathbf{v}_n.$$

The only distributive property you can take for granted is that $a(b + c) = ab + ac$ for $a, b, c \in \mathbb{R}$.

4. (10 points) Let A be a 4×3 matrix, and let $\mathbf{b}$ and $\mathbf{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about the number of solutions of the system $A\mathbf{x} = \mathbf{c}$? In other words, is it possible for $A\mathbf{x} = \mathbf{c}$ to have no solutions? exactly one solution? infinitely many solutions?

5. (10 points) Consider the system of linear equations with augmented matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 5 & -1 & 3 \\ -1 & -2 & 0 & -3 & 1 & -2 \\ -2 & -4 & -1 & -8 & 2 & -5 \end{bmatrix}.$$

Show that there exist three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^5$ with the property that any solution $\mathbf{x} \in \mathbb{R}^5$ of this system can be written as

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. (Be sure to explain why the vectors you find are indeed linearly independent!)