Math 291-1: Midterm 1 Solutions
Northwestern University, Fall 2015

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) A linear system of 3 equations with 2 variables cannot have infinitely many solutions.

(b) If A, B are 2×2 matrices such that the set solutions of $A\mathbf{x} = \mathbf{0}$ is spanned by $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and the set of solutions of $B\mathbf{x} = \mathbf{0}$ is spanned by $\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right)$, then A and B have the same reduced row-echelon form.

Solution. (a) This is false. For instance, the linear system

$$\begin{aligned} x + y &= 0 \\ x + y &= 0 \\ x + y &= 0 \end{aligned}$$

has infinitely many solutions. More generally, any system whose augmented matrix has a reduced echelon form with a single pivot will serve as a counterexample.

(b) This is true. Note that since $\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right)$ are both multiples of $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ is a multiple of either of these, the set of solutions of both $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are spanned by $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and hence these equations have the same solutions, which implies A and B have the same reduced echelon form. $\square$

2. Consider the linear system with augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 & -1 & | & 0 \\ 2 & -2 & 3 & -5 & | & -1 \\ -3 & 3 & -6 & 12 & | & 3 \end{pmatrix}.$$

The vector

$$\mathbf{x}_0 = \begin{pmatrix} -3 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

gives a solution. Find two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^4$ with the property that any solution of the system above can be written as

$$\mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some $c_1, c_2 \in \mathbb{R}$. Justify the reason why you're claimed vectors work.

Solution. Let

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 3 & -5 \\ -3 & 3 & -6 & 12 \end{pmatrix}$$

and consider the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$. Row reducing the augmented matrix of the homogeneous equation gives:

$$\begin{pmatrix} 1 & -1 & 1 & -1 & | & 0 \\ 2 & -2 & 3 & -5 & | & -1 \\ -3 & 3 & -6 & 12 & | & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -3 & | & 0 \\ 0 & 0 & -3 & 9 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 2 & | & 0 \\ 0 & 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Thus the solutions of $A\mathbf{x} = \mathbf{0}$ are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - 2x_4 \\ x_2 \\ 3x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix}.$$

Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix}.$$

The solutions of an inhomogeneous equation $A\mathbf{x} = \mathbf{b}$ are obtained by adding to a particular solution the solutions of $A\mathbf{x} = \mathbf{0}$ (as we showed in the Warm-Up of Lecture 12), so we conclude that any solution of the given system is of the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

as desired. □

3. Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly independent but that for some $\mathbf{w} \in \mathbb{R}^n$, the vectors

$$\mathbf{v}_1 + \mathbf{w}, \ldots, \mathbf{v}_k + \mathbf{w}$$

obtained by adding $\mathbf{w}$ to each $\mathbf{v}_i$ are linearly dependent. Show that $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

Proof. Since $\mathbf{v}_1 + \mathbf{w}, \ldots, \mathbf{v}_k + \mathbf{w}$ are linearly dependent there exist scalars $a_1, \ldots, a_k \in \mathbb{R}$, at least one of which is nonzero, such that

$$a_1(\mathbf{v}_1 + \mathbf{w}) + \cdots + a_k(\mathbf{v}_k + \mathbf{w}) = \mathbf{0}.$$

Rearranging this gives

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = -(a_1 + \cdots + a_k)\mathbf{w}.$$

If $a_1 + \cdots + a_k = 0$, this equation becomes $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$, which, since at least one a_i is nonzero, contradicts the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Hence we must have $a_1 + \cdots + a_k \neq 0$, so we get

$$\mathbf{w} = -\frac{a_1}{a_1 + \cdots + a_k}\mathbf{v}_1 - \cdots - \frac{a_k}{a_1 + \cdots + a_k}\mathbf{v}_k$$

and thus $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ as claimed. □

4. Prove that

$$a(\mathbf{v}_1 + \cdots + \mathbf{v}_n) = a\mathbf{v}_1 + \cdots + a\mathbf{v}_n$$

for any *complex* scalar $a \in \mathbb{C}$ and $n \geq 2$ complex vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{C}^2$. You cannot take it for granted that multiplication of complex numbers is distributive; you must prove this if you need it.

Proof. First we show that

$$a(z + w) = az + aw$$

holds for complete numbers $a, z, w \in \mathbb{C}$. Write each of these in terms of their real and imaginary parts as:

$$a = c + id, \; z = p + iq, \; w = m + in$$

where $c, d, p, q, m, n \in \mathbb{R}$. Then:

$$a(z+w) = (c+id)[(p+m)+i(q+n)] = c(p+m) - d(q+n) + i[c(q+n)+d(p+m)]$$

and

$$az + aw = (c+id)(p+iq) + (c+id)(m+in) = (cp-dq) + i(cq+dp) + (cm+nd) + i(cn+md).$$

Comparing these last two expressions we see that they are equal, so

$$a(z+w) = az + aw$$

as claimed.

We now proceed by induction. For two vectors

$$\mathbf{v}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \ \mathbf{v}_w = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix} \in \mathbb{C}^2,$$

and a scalar $a \in \mathbb{C}$, we have

$$a(\mathbf{v}_1 + \mathbf{v}_2) = a\begin{pmatrix} z_1 + z_2 \\ w_1 + w_2 \end{pmatrix} = \begin{pmatrix} a(z_1+z_2) \\ a(w_1+w_2) \end{pmatrix} = \begin{pmatrix} az_1 + az_2 \\ aw_1 + aw_2 \end{pmatrix} = \begin{pmatrix} az_1 \\ aw_1 \end{pmatrix} + \begin{pmatrix} az_2 \\ aw_2 \end{pmatrix} = a\mathbf{v}_1 + a\mathbf{v}_2,$$

so the claimed equality holds for the base case of $n = 2$. Suppose it holds for any k vectors and let $\mathbf{v}_1, \ldots, v_{k+1}$ be any $k+1$ vectors in $\mathbb{C}^2$. Then

$$\begin{aligned} a(\mathbf{v}_1 + \cdots + \mathbf{v}_{k+1}) &= a([\mathbf{v}_1 + \cdots + \mathbf{v}_k] + \mathbf{v}_{k+1}) \\ &= a(\mathbf{v}_1 + \cdots + \mathbf{v}_k) + a\mathbf{v}_{k+1} \\ &= a\mathbf{v}_1 + \cdots + a\mathbf{v}_k + a\mathbf{v}_{k+1} \end{aligned}$$

where in the second line we have used the base case and in the third the induction hypothesis. We conclude by induction that

$$a(\mathbf{v}_! + \cdots + \mathbf{v}_n) = a\mathbf{v}_1 + \cdots + a\mathbf{v}_n$$

for any $n \geq 2$ vectors. □

5. Suppose that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$ span $\mathbb{R}^4$. Let A be the 4×4 matrix having $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as columns. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$ are vectors such that $A\mathbf{x} = A\mathbf{y}$, show that $\mathbf{x} = \mathbf{y}$. Hint: Of which equation is $\mathbf{x} - \mathbf{y}$ a solution?

Proof. Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^4$, the row reduced echelon form of A must be the identity matrix, which in turn implies that the only solution of $A\mathbf{z} = \mathbf{0}$ is $\mathbf{z} = \mathbf{0}$. If $A\mathbf{x} = A\mathbf{y}$, then $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ so $\mathbf{x} - \mathbf{y}$ is a solution of $A\mathbf{z} = \mathbf{0}$. Since the only solution of this is $\mathbf{z} = \mathbf{0}$, we must have $\mathbf{x} - \mathbf{y} = \mathbf{0}$ so $\mathbf{x} = \mathbf{y}$ as claimed.

Phrased in terms of newer language, this says that if the columns of a 4×4 matrix A span $\mathbb{R}^4$, A defines an injective transformation, which is simply a part of the Amazingly Awesome Theorem. □