Math 291-1: Midterm 1 Solutions
Northwestern University, Fall 2016

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample. (A counterexample is a specific example in which the given claim is indeed false.)

(a) If $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{C}^2$ are complex vectors which are linearly dependent over $\mathbb{C}$, then $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are also linearly dependent over $\mathbb{R}$. (Recall that the distinction between "over $\mathbb{C}$" and "over $\mathbb{R}$" is whether or not we allow arbitrary complex scalars or only real scalars as coefficients.)

(b) If A, B are matrices for which $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solutions, then $A = B$.

Solution. (a) This is false. For instance, the vectors

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{w}_2 = \begin{bmatrix} i \\ 0 \end{bmatrix}, \ \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are linearly dependent over $\mathbb{C}$ since

$$\mathbf{w}_2 = i\mathbf{w}_1 + 0\mathbf{w}_3.$$

However, these are linearly independent over $\mathbb{R}$. Indeed, suppose

$$a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} i \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$. Then

$$a + bi = 0 \text{ and } c = 0.$$

Since $a, b \in \mathbb{R}$, the only way for $a + bi$ to be 0 is for a and b to both be 0, so we get $a = b = c = 0$, and hence $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent over $\mathbb{R}$.

(b) This is false. Take for instance

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then the only solution of both $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, and yet $A \neq B$. The point is that the given condition implies that A and B are row equivalent, but certainly row equivalent matrices do not have to be equal. □

2. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$. Show that $\mathbf{b} \in \mathbb{R}^3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ if and only if $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_2$. (This shows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_2$ have the same span.)

Bonus (2 extra points): Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Show that $\mathbf{b} \in \mathbb{R}^n$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ if and only if $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \ldots, \mathbf{v}_k - \mathbf{v}_{k-1}$, where each vector in this new list except the first is of the form $\mathbf{v}_i - \mathbf{v}_{i-1}$ for $i = 2, \ldots, k$. Note that the original problem is a special case of the Bonus, so doing the Bonus alone will get you the full 12 points.

Proof. We'll give the proof of the Bonus only, since the stated problem is the special case of the bonus when $n = 3$. First suppose $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, so there exist $a_1, \ldots, a_k \in \mathbb{R}$ such that

$$\mathbf{b} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k.$$

We can rewrite this as:

$\mathbf{b} = (a_1 + \cdots + a_k)\mathbf{v}_1 + (a_2 + \cdots + a_k)(\mathbf{v}_2 - \mathbf{v}_1) + \cdots + (a_{k-1} + a_k)(\mathbf{v}_{k-1} - \mathbf{v}_{k-2}) + a_k(\mathbf{v}_k - \mathbf{v}_{k-1}),$

which shows that $\mathbf{b}$ is a linear combination of

$$\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_2, \ldots, \mathbf{v}_k - \mathbf{v}_{k-1}$$

as desired. (Note that the coefficients above were found by setting

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2(\mathbf{v}_2 - \mathbf{v}_1) + \cdots + c_k(\mathbf{v}_k - \mathbf{v}_{k-1}),$$

and rearranging to get

$$\mathbf{b} = (c_1 - c_2)\mathbf{v}_1 + (c_2 - c_3)\mathbf{v}_2 + \cdots + (c_{k-1} - c_k)\mathbf{v}_{k-1} + c_k\mathbf{v}_k.$$

Comparing this with the original expression for $\mathbf{b}$, we are thus looking for scalars $c_1, \ldots, c_k$ satisfying

$$a_1 = c_1 - c_2,\ a_2 = c_2 - c_3, \ldots,\ a_{k-1} = c_{k-1} - c_k,\ a_k = c_k.$$

Starting from the end, this gives

$$c_k = a_k,\ c_{k-1} = a_{k-1} + c_k = a_{k-1} + a_k,\ c_{k-2} = a_{k-2} + c_{k-1} = a_{k-2} + a_{k-1} + a_k,$$

and so on until $c_1 = a_1 + \cdots + a_k$.)

Conversely suppose $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \ldots, \mathbf{v}_k - \mathbf{v}_{k-1}$:

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2(\mathbf{v}_2 - \mathbf{v}_1) + \cdots + c_k(\mathbf{v}_k - \mathbf{v}_{k-1}).$$

Rearranging terms gives:

$$\mathbf{b} = (c_1 - c_2)\mathbf{v}_1 + (c_2 - c_3)\mathbf{v}_2 + \cdots + (c_{k-1} + c_k)\mathbf{v}_{k-1} + c_k\mathbf{v}_k,$$

so $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ as desired. $\square$

3. Consider the system of linear equations in four variables with augmented matrix

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 3 & 3 \\ -3 & -9 & -4 & -5 & -7 \\ 2 & 6 & 4 & 6 & 6 \end{bmatrix}.$$

Show that any solution $\mathbf{x} \in \mathbb{R}^4$ of this system can be written as

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

for some $a, b \in \mathbb{R}$. Hint: You can use the result of the Bonus in Problem 2 without justification.

Proof. Reducing the given augmented matrix via row operations gives:

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 3 & 3 \\ -3 & -9 & -4 & -5 & -7 \\ 2 & 6 & 4 & 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 2 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the general solution is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - 3x_2 + x_4 \\ x_2 \\ 1 - 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Now, by the Bonus on Problem 2, any vectors which is a linear combination of

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

is also a linear combination of

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \\ 1 \end{bmatrix}.$$

Thus the portion of the general solution which looks like

$$x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

can be rewritten as

$$a \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

for some $a, b \in \mathbb{R}$, so the general solution is also of the form

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

as claimed. □

4. Let A be a 2×2 matrix. Prove that

$$A(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n) = c_1 A\mathbf{x}_1 + \cdots + c_n A\mathbf{x}_n$$

for any $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^2$ and any $c_1, \ldots, c_n \in \mathbb{R}$. You cannot take $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ nor $A(c\mathbf{x}) = cA\mathbf{x}$ for granted, and must justify these facts first if you need them.

Proof. Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
Then for $\mathbf{x} = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ and $\mathbf{y} = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ we have:
$$A(\mathbf{x}+\mathbf{y}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + ay_1 + bx_2 + by_2 \\ cx_1 + cy_1 + dx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} + \begin{bmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{bmatrix} = A\mathbf{x} + A\mathbf{y}.$$
Also, if $r \in \mathbb{R}$, we have:
$$A(r\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix} = \begin{bmatrix} arx_1 + brx_2 \\ crx_1 + drx_2 \end{bmatrix} = r \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = rA\mathbf{x}.$$

Now, using the facts above, we see that for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ and $c_1, c_2 \in \mathbb{R}$ we have:
$$A(c_1\mathbf{x}_1 + c_2\mathbf{x}_3) = A(c_1\mathbf{x}_1) + A(c_2\mathbf{x}_2) = c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2.$$
Hence the required property holds in the base case of two vectors. Suppose it holds for any number of $n-1 \geq 2$ vectors and let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^2$ and $c_1, \ldots, c_n \in \mathbb{R}$. Then
$$\begin{aligned} A(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n) &= A([c_1\mathbf{x}_1 + \cdots + c_{n-1}\mathbf{x}_{n-1}] + c_n\mathbf{x}_n) \\ &= A(c_1\mathbf{x}_1 + \cdots + c_{n-1}\mathbf{x}_{n-1}) + c_nA\mathbf{x}_n \\ &= c_1A\mathbf{x}_1 + \cdots + c_nA\mathbf{x}_n, \end{aligned}$$
where in the second line we use the base case and in the third line we use the induction hypothesis. We conclude by induction that the claimed property indeed holds for any number of vectors. □

5. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ and let A be the matrix with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns. If there exists $\mathbf{b} \in \mathbb{R}^3$ for which $A\mathbf{x} = \mathbf{b}$ has no solution, show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

Proof. Consider the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$. If the corresponding system has no solution, then the reduced form of this augmented matrix must have a row of the form
$$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$
in order for there to be a row corresponding to the impossible equation
$$0 = 1.$$
This says that the reduced form of A itself must have at least one row of all zeroes. Now, reducing the augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ gives
$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} \operatorname{rref}(A) & \mathbf{0} \end{bmatrix}.$$
Since $\operatorname{rref}(A)$ has a row of all zeroes, this reduce augmented matrix must have a column without a pivot since there are only at most 2 pivots. This implies that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, which implies that the columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of A are linearly dependent. □