Math 291-3: Midterm 1 Solutions
Northwestern University, Spring 2016

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) Any Riemann sum of the function $f(x, y) = x$ over $[-1, 1] \times [-1, 1]$ is positive.

(b) The function

$$f(x,y) = \begin{cases} x^2 + y^2 & \|(x,y)\| \leq 1 \\ \frac{2}{x^2+y^2} & \|(x,y)\| > 1 \end{cases}$$

is integrable over the square $[-1, 1] \times [-1, 1]$.

Solution. (a) This is false. For one counterexample, take the partition consisting of all of $[-1, 1] \times [-1, 1]$ itself, meaning that we do not divide it into smaller rectangles at all. Take $(-1, -1)$ as a sample point. Then the Riemann sum of f corresponding to this partition and this sample point is

$$f(-1, -1)\,\text{area}([-1, 1] \times [-1, 1]) = -4,$$

which is not positive.

(b) This is true. First, the function $x^2 + y^2$ is continuous on the compact set $\|(x, y)\| \leq 1$, so it is bounded. Also, $\frac{2}{x^2+y^2}$ is continuous on the compact set consisting of the part of the region $\|(x, y)\| \geq 1$ within the square in question, so it is also bounded. Thus f is bounded over both regions $\|(x, y)\| \leq 1$ and $\|(x, y)\| > 1$, so it is bounded over the entire square. Moreover, f fails to be continuous only on the unit circle $\|(x, y)\| = 1$, so since this has measure zero, we conclude that f is integrable over the square $[-1, 1] \times [-1, 1]$. □

2. An n-sided polygon with side lengths $x_1, \ldots, x_n$ has area — incorrect. Ignore interpretation as an area

$$A = \sqrt{(s - x_1)(s - x_2) \cdots (s - x_n)}$$

where s is half the perimeter. Show that among all n-sided polygons with fixed perimeter P, there is one with maximal area and determine the lengths of all its sides.

Proof. First we show that there is an n-sided polygon of perimeter P with maximal area. Since $x_1 + \cdots + x_n = P$ and each x_i is nonnegative, each x_i must be between 0 and P, so the region in $\mathbb{R}^n$:

$$E := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ and } x_1 + \cdots + x_n = P\}$$

consisting of points $(x_1, \ldots, x_n)$ satisfying the constraints $x_i \geq 0$ and $x_1 + \cdots + x_n = P$ lies within the box $[0, P] \times \cdots \times [0, P]$ and hence is bounded. This region is also closed since it is defined by an equality $x_1 + \cdots + x_n = P$, so it is compact. The function

$$A(x_1, \ldots, x_n) = \sqrt{s(s - x_1) \cdots (s - x_n)}$$

is continuous on E, and hence by the Extreme Value Theorem it has a maximum and a minimum. Note that A is maximized when $f = A^2$ is maximized.

Now, by the method of Lagrange multipliers, the values $(x_1, \cdots, x_n)$ which maximize $f = A^2$ among points in E satisfy

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$$

for some λ where $g(\mathbf{x}) = x_1 + \cdots + x_n$. This gives

$$(-s(s-x_2)\cdots(s-x_n), \cdots, -s(s-x_1)\cdots(s-x_{n-1})) = \lambda(1, \cdots, 1),$$

so $x_1, \cdots, x_n$ satisfy

$$\begin{aligned} -s(s-x_2)\cdots(s-x_n) &= \lambda \\ -s(s-x_1)(s-x_3)\cdots(s-x_n) &= \lambda \\ -s(s-x_2)\cdots(s-x_{n-1}) &= \lambda \\ x_1 + \cdots + x_n &= P. \end{aligned}$$

Since $P > 0$, $s = \frac{1}{2}(x_1 + \cdots + x_n) = \frac{1}{2}P > 0$ so the first two equations give the requirement that

$$(s-x_2)(s-x_3)\cdots(s-x_n) = (s-x_1)(s-x_3)\cdots(s-x_n).$$

If $x_i = s$ for any i, then $A(\mathbf{x}) = 0$, which is not the maximum we are looking for since, for instance, taking $x_1 = \cdots = x_n = \frac{1}{n}P$ gives a positive value for A. Thus we may assume each $x_i \neq s$. Then the equation above gives

$$s - x_2 = s - x_1, \text{ so } x_1 = x_2.$$

A similar argument using the other pairs of equations among the original ones give $x_i = x_j$ for all $i \neq j$. The constraint gives $x_1 = \cdots = x_n = \frac{1}{n}P$, which give the maximum area as claimed. (The minimum values of A are the ones where one of the x_i equals s.) □

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = -x^2 - y^3 + 3y$ and let $D \subseteq \mathbb{R}^2$ be the region enclosed by the circle $x^2 + y^2 = 2y$. Show that

$$\iint_D f(x, y)\, dA \leq 2\pi.$$

Hint: You can take it for granted without justification that the maximum value of f over D does not occur on the boundary of D.

Proof. By the hint, the maximum value of f occurs within D and not on its boundary. Since

$$\nabla f = (-2x, -3y^2 + 3),$$

the critical points of f are at $(0, \pm 1)$, and only $(0, 1)$ lies within D. You can check that this is indeed a maximum by checking that the Hessian at $(0, 1)$ is negative definite. Since $f(0, 1) = 2$, the maximum value of f in D is 2. Thus

$$\iint_D f(x, y)\, dA \leq \iint_D 2\, dA = 2\, \text{area}(D).$$

The given circle can be written as $x^2 + (y-1)^2 = 1$, so D is a disk of radius 1 and thus has area π. Hence we get $\iint_D f(x, y)\, dA \leq 2\pi$ as claimed. □

4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Rewrite the following as an iterated integral with respect to the order $dy\, dz\, dx$.

$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\, dz\, dx\, dy.$$

Solution. The region of integration is bounded by the xy-plane, the yz-plane, the plane $x + y = 1$, and the surface $z = y^2$. The shadow of this in the xz-plane is the portion of the first quadrant lying below the curve obtained by pushing the intersection of the plane $x + y = 1$ with the surface $z = y^2$ onto the xz-plane. This curve has equation

$$z = (1 - x)^2,$$

which we find by eliminating y from $x + y = 1$ and $z = y^2$. Thus in the new order, the bounds on x are 0 to 1 and the bounds on z are 0 to $(1 - x)^2$. Finally, at a fixed (x, z), the values of y within the region of integration start on the left along $z = y^2$ and move to the right to $x + y = 1$, so the bounds on y are $\sqrt{z}$ to $1 - x$. Thus the given integral in the rewritten order is

$$\int_0^1 \int_0^{(1-x)^2} \int_{\sqrt{z}}^{1-x} f(x, y, z)\, dy\, dz\, dx.$$

$\square$

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^1 (1 - x) f(x)\, dx = 5.$$

Find the value of the double integral

$$\int_0^1 \int_0^x f(x - y)\, dy\, dx.$$

Hint: Let $u = x - y$ and use this as one of the new variables in a suitable change of variables application.

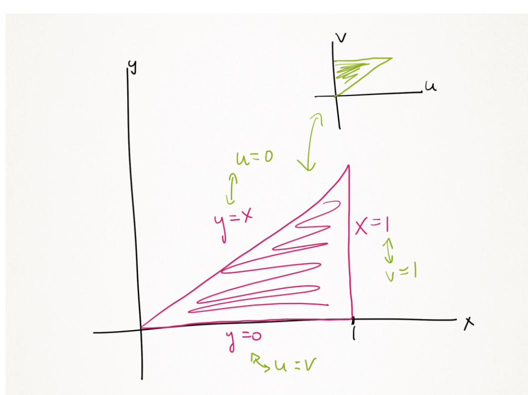
Solution. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the function

$$\phi(u, v) = (v, v - u),$$

so $x = v$ and $y = v - u$. This is C^1 and one-to-one and has Jacobian matrix

$$D\phi = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

everywhere. For the region E in the uv-plane bounded by $u = 0, u = v$, and $v = 1$ we have that $\phi(E)$ is described by the bounds on the given double integral:



Thus the change of variables formula gives:

$$\int_0^1 \int_0^x f(x-y)\,dy\,dx = \iint_{\phi(E)} f(x-y)\,d(x,y) = \iint_E f(u)|\det D\phi(u,v)|\,d(u,v) = \iint_E f(u)\,d(u,v)$$

since $\det D\phi = 1$. By Fubini's Theorem the resulting integral is:

$$\iint_E f(u)\,d(u,v) = \int_0^1 \int_u^1 f(u)\,dv\,du = \int_0^1 (1-u)f(u)\,du = 5$$

by the assumption that $\int_0^1 (1-x)f(x)\,dx = 5$. $\square$