Math 291-3: Midterm 1 Solutions
Northwestern University, Spring 2017

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If $f : [-1,1] \times [-2,2] \times [-3,3] \to \mathbb{R}$ is a constant function, then all Riemann sums of f (for any partition of $[-1,1] \times [-2,2] \times [-3,3]$ and any collection of sample points) have the same value.

(b) If $f : [-5,5] \times [-5,5] \to \mathbb{R}$ is bounded but not continuous, then f is not integrable.

Solution. (a) This is true. Say that $f(\mathbf{x}) = M$ for all $\mathbf{x}$ and let P be any partition of the given box and let $\mathbf{c}_i$ be any collection of sample points. Then letting B_i denote the smaller boxes determined by the partition P, we have:

$$\begin{aligned} R(f, P, \mathbf{c}_i) &= \sum_i f(\mathbf{c}_i) \operatorname{Vol}(B_i) \\ &= \sum_i M \operatorname{Vol}(B_i) \\ &= M \sum_i \operatorname{Vol}(B_i) \\ &= M \operatorname{Vol}([-1,1] \times [-2,2] \times [-3,3]) \end{aligned}$$

where in the third step we can pull out M since it is a constant, and in the last step we use the fact that adding together the volumes of all the B_i gives the volume of the original larger box.

(b) This is false. For example, the function which is 1 everywhere except at $(0,0)$, where it is 2, is bounded and not continuous at $(0,0)$, but it is integrable since it only fails to be continuous at a single point, which has measure zero in $\mathbb{R}^2$. $\square$

2. Fix $K > 0$ and consider all nonnegative numbers $x_1, \dots, x_n$ satisfying

$$x_1 + x_2 + \cdots + x_n = K.$$

Show that among all such numbers there exists ones which maximize the product $x_1 x_2 \cdots x_n$ and find the specific values of those which do.

Proof. First, the functions $f(\mathbf{x}) = x_1 \cdots x_n$ is continuous on the constraint set, which is compact. (It is compact because it closed and bounded, since none of the x_i can be larger than K and still satisfy the constraint.) Thus f has a maximum value over the constraint set by the Extreme Value Theorem. Note also that taking each $x_i = \frac{K}{n}$ gives points satisfying the constraint at which f is positive, so the maximum value of f over the constraint set must be positive as well.

By the method of Lagrange Multipliers, the points which give this maximum value are among those which satisfy

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$$

for some $\lambda \in \mathbb{R}$, where $g(\mathbf{x}) = x_1 + \cdots + x_n$ is the function defining the constraint. This equation becomes

$$\langle x_2 x_3 \cdots x_n, \dots, x_1 x_2 \cdots x_{n-1} \rangle = \lambda \langle 1, \dots, 1 \rangle,$$

which after comparing components turns into the condition that

$$x_2 x_3 \cdots x_n = x_1 x_3 \cdots x_n = x_1 x_2 x_4 \cdots x_n = \cdots = x_1 x_2 \cdots x_{n-1}.$$

We may assume that none of the x_i are zero since otherwise $f(\mathbf{x}) = 0$ and we know that 0 is not the maximum value of f for which we are looking. Thus we may divide all expressions above by various variables to get that

$$x_1 = x_2 = \cdots = x_n$$

in the end. Thus the maximum of f on the constraint set is attained when all x_i are the same (this is not a minimum since the minimum is 0 when some x_i is zero), and the value of these variables according to the constraint is then $x_i = \frac{K}{n}$ for all i. $\square$

3. Show that for any compact region $D \subseteq \mathbb{R}^2$ of area 10, the following inequality holds:

$$\iint_D (3 - x^2 + 2x - y^2 + 2y)\, dA \le 50.$$

You may assume that any local maximum of $f(x, y) = 3 - x^2 + 2x - y^2 + 2y$ is actually a global maximum.

Proof. We first find the local, and hence global, maximum of f within D. Setting $\nabla f = 0$ gives

$$\langle -2x + 2, -2y + 2 \rangle = \langle 0, 0 \rangle,$$

so $x = 1$ and $y = 1$. The Hessian of f at $(1, 1)$ is

$$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

and since this is negative definite we know that $(1, 1)$ is a local maximum of f. Thus the maximum value of f over all of D is $f(1, 1) = 5$. Hence since $f(x, y) \ge 5$ for all $(x, y) \in D$, we have

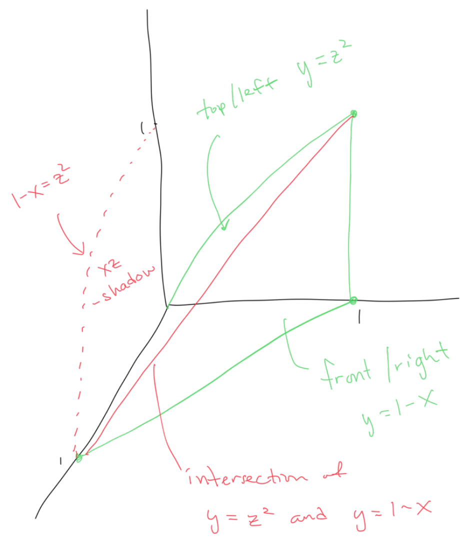
$$\iint_D (3 - x^2 + 2x - y^2 + 2y)\, dA \le \iint_D 5\, dA = 5 \iint_D dA = 5\,\mathrm{Area}(D),$$

and the claim follows since $\mathrm{Area}(D) = 10$. $\square$

4. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous. Rewrite the following as an iterated integral with respect to the order $dy\, dx\, dz$.

$$\int_0^1 \int_{z^2}^1 \int_0^{1-y} f(x, y, z)\, dx\, dy\, dz$$

Solution. The region of integration E in the given integral looks like



Indeed, the shadow in the yz-plane lies above $z = 0$, below $y = z^2$, and to the left of $y = 1$ according the given bounds on z and y, and then at a fixed (y, z), the values for x start on the yz-plane at $x = 0$ and move out forward as far as the plane $x = 1 - y$. The top/left of E is given by the surface $y = z^2$, the front/right by the plane $x = 1 - y$, the bottom by the xy-plane, and the back by the yz-plane.

The shadow of E in the xz-plane is drawn on the left in the picture above. The curve $1 - x = z^2$ lies directly to the left of the curve in E formed by intersecting the surface $y = z^2$ with the plane $x = 1 - y$, and its equation is found by eliminating y in these two equations. Thus, with respect to $dy\,dx\,dz$, z goes from 0 to 1, and at a fixed z the value of x goes from $x = 0$ to $x = 1 - z^2$. Then, at a fixed (x, z), the value of y in E begins on the left at $y = z^2$ and moves to the right until $y = 1 - x$. Hence the given integral becomes

$$\int_0^1 \int_0^{1-z^2} \int_{z^2}^{1-x} f(x, y, z)\, dy\, dx\, dz.$$

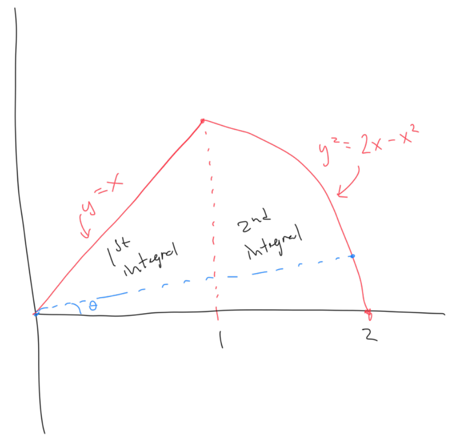
$\square$

5. Write the following as a single iterated integral in polar coordinates.

$$\int_0^1 \int_y^1 (x^2 + y^2)\, dx\, dy + \int_1^2 \int_0^{\sqrt{2x - x^2}} (x^2 + y^2)\, dy\, dx$$

Note that the order of integration in the first expression is $dx\,dy$ while in the second it is $dy\,dx$.

Solution. The first set of bounds describes the triangular region in the xy-plane lying below $y = x$, above $y = 0$ and to the left of $x = 1$. The second set of bounds then covers x values from 1 to 2, with y moving at a fixed x from $y = 0$ up to the curve $y = \sqrt{2x - x^2}$. After squaring both sides and completing the square, this latter curve becomes $(x - 1)^2 + y^2 = 1$, so it describes a circle of radius 1 centered at $(1, 0)$. This forms the right boundary of the region in question, so the combined region looks like:



Now, in polar coordinates, this region is given by θ values going from 0 to $\frac{\pi}{4}$, and at any fixed θ the value of r moves from $r = 0$ at the origin out towards the given circle. Writing the equation of this circle as $x^2 + y^2 = 2x$, we see that in polar coordinates this becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Hence r moves from 0 to $2\cos\theta$, so the integral in polar coordinates is:

$$\int_0^{\frac{\pi}{4}} \int_0^{2\cos\theta} r^3 \, dr \, d\theta,$$

where one factor of r in the integrand comes from the Jacobian factor and two factors from converting $x^2 + y^2$. $\square$