Math 291-2: Midterm 1 Solutions
Northwestern University, Winter 2018

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If $\mathbf{v}_1, \mathbf{v}_2$ is a basis of $\mathbb{R}^2$ and $\mathbf{x} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, then $\mathbf{x} = \operatorname{proj}_{\mathbf{v}_1} \mathbf{x} + \operatorname{proj}_{\mathbf{v}_2} \mathbf{x}$.

(b) If A is a 2×2 matrix which sends a disk of radius 2 onto a disk of radius 1, then $|\det A| < 1$.

Solution. (a) This is false, and is in fact only true for an orthogonal basis. For a counterexample take $\mathbf{v}_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathbf{v}_2 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. Then

$$\operatorname{proj}_{\mathbf{v}_1} \mathbf{x} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \text{ and } \operatorname{proj}_{\mathbf{v}_2} \mathbf{x} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right],$$

which do not add up to $\mathbf{x}$.

(b) This is true. The expansion factor interpretation of $|\det A|$ says that

$$(\text{area of image disk}) = |\det A|(\text{area of original disk}),$$

so $\pi = |\det A|(4\pi)$ and hence $|\det A| = \frac{1}{4} < 1$. □

2. Let A be an $n\times n$ symmetric matrix and let V be a subspace of $\mathbb{R}^n$ with the property that $A\mathbf{v} \in V$ for any $\mathbf{v} \in V$. Show that if $\mathbf{w} \in V^\perp$, then $A\mathbf{w} \in V^\perp$.

Proof. (This was on the first homework.) Let $\mathbf{w} \in V^\perp$. Then for any $\mathbf{v} \in V$ we have:

$$A\mathbf{w}\cdot\mathbf{v} = \mathbf{w}\cdot A\mathbf{v} = 0$$

where the first equality follows from the fact that A is symmetric and the second from the fact that $A\mathbf{v} \in v$ and $\mathbf{w}$ is orthogonal to everything in V. Thus $A\mathbf{w}$ is orthogonal to everything in V, so $A\mathbf{w} \in V^\perp$. □

3. Suppose A and B are $n\times n$ orthogonal matrices such that AB^T is upper triangular with positive diagonal entries. Show that $A = B$. Hint: The product of orthogonal matrices is orthogonal.

Proof. First, since B is orthogonal B^T is orthogonal as well, and thus AB^T is orthogonal. Say that AB^T looks like

$$\begin{bmatrix} a_1 & * & \cdots & * \\ & a_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & a_n \end{bmatrix}$$

where $a_1, \ldots, a_n$ are all positive and blanks denote zeroes. If this is orthogonal, the first column must have length 1, so $a_1 = \pm 1$ and hence $a_1 = 1$ since this entry should be positive. Next the second column must be orthogonal to the first, which implies that the entry above a_2 must be 0, so the second column looks like

$$\begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

But then this should have length 1, so $a_2 = \pm 1$ and thus $a_2 = 1$ since this should be positive. In general, if we've already shown that the first k columns are simply $\mathbf{e}_1, \dots, \mathbf{e}_k$, then the $(k+1)$-st column must look like

$$\begin{bmatrix} 0 \\ \vdots \\ a_{k+1} \\ \vdots \\ 0 \end{bmatrix}$$

in order for this to be orthogonal to the previous $\mathbf{e}_1, \dots, \mathbf{e}_k$. As before, the fact that this has length 1 with $a_{k+1} > 0$ implies that this column is $\mathbf{e}_{k+1}$, so we conclude that $AB^T = I$ is the identity matrix. Multiplying by B on both sides gives $AB^TB = B$, so $A = B$ since $B^TB = I$ because B is orthogonal. □

4. Suppose A, B are $n \times n$ matrices. Show that $\det(AB) = (\det A)(\det B)$. Hint: In the case where A is invertible, consider what happens when you row-reduce the matrix $\begin{bmatrix} A & AB \end{bmatrix}$ to turn the A on the left into I.

Proof. First, if A is non-invertible, then AB is also non-invertible so $\det A$ and $\det(AB)$ are both 0 in this case, so $\det(AB) = (\det A)(\det B)$ is true.

Suppose now that A is invertible, so that A is row-reducible to the identity. This reduction gives

$$\det I = (-1)^k c_\ell \cdots c_1 (\det A)$$

where k is the number of row swaps used in the reduction and the c_i are the nonzero scalars used in any operations which scale a row by a nonzero value. (Recall that adding a multiple of one row to another does not affect the determinant.) This gives

$$\det A = \frac{1}{(-1)^k c_\ell \cdots c_1}.$$

Now, the operations which transform A into I will also transform AB into B:

$$\begin{bmatrix} A & AB \end{bmatrix} \to \begin{bmatrix} I & B \end{bmatrix}$$

since they amount to multiplying A by A^{-1} on the left, so we get

$$\det B = (-1)^k c_\ell \cdots c_1 (\det AB)$$

for the same k and c_i as before. Thus

$$\det(AB) = \frac{1}{(-1)^k c_\ell \cdots c_1} (\det B) = (\det A)(\det B)$$

as required. □

5. Let $T : P_5(\mathbb{R}) \to P_5(\mathbb{R})$ be the linear transformation defined by

$$T(p(x)) = 2x^2 p''(x).$$

Determine all eigenvalues and eigenvectors of T. Be sure to justify why the eigenvalues and eigenvectors you find are indeed all of them.

Solution. First note that for any $0 \leq k \leq 5$, we have:

$$T(x^k) = 2x^2[k(k-1)x^{k-2}] = 2k(k-1)x^k.$$

This shows that each x^k is an eigenvector of T with eigenvalue $2k(k-1)$, so we (so far) get eigenvalues

$$0, 0, 4, 12, 24, 40$$

with eigenvectors

$$1, x, x^2, x^3, x^4, x^5$$

respectively. Now, this so far gives the following information about the geometric multiplicities:

$$\dim E_0 \geq 2,\ \dim E_4 \geq 1,\ \dim E_{12} \geq 1,\ \dim E_{24} \geq 1,\ \dim E_{40} \geq 1.$$

Since this lower bounds already add up to $\dim P_5(\mathbb{R}) = 6$, there can be no further eigenvalues and these lower bounds must in fact equal the given dimensions. Thus we see that

$$E_0 = \operatorname{span}\{1, x\},\ E_4 = \operatorname{span}\{x^2\},\ E_{12} = \operatorname{span}\{x^3\},\ E_{24} = \operatorname{span}\{x^4\},\ E_{40} = \operatorname{span}\{x^5\}$$

describe all eigenspaces explicitly. $\square$