Math 291-1: Midterm 2
Northwestern University, Fall 2016

Name: ______________________________

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample. (A counterexample is a specific example in which the given claim is indeed false.)

(a) If $A \in M_2(\mathbb{R})$ describes reflection across a line passing through the origin, then A is invertible.

(b) The space $M_4(\mathbb{R})$ does not have a 5-dimensional subspace.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. On the board is a proof that if A is a 2×2 matrix and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are linearly independent vectors such that

$$A\mathbf{v}_1 = \mathbf{0} \text{ and } A\mathbf{v}_2 \in \operatorname{span}(\mathbf{v}_1),$$

then $A^2 = 0$. Using this as a guide, prove that if A is an $n \times n$ matrix and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ are linearly independent vectors such that

$$A\mathbf{v}_1 = \mathbf{0} \text{ and } A\mathbf{v}_k \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) \text{ for } k = 2, \ldots, n,$$

then $A^n = 0$.

3. (10 points) Suppose A is an $n \times n$ matrix and that $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ form a basis of $\mathbb{R}^n$. Show that A is invertible if and only if $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ form a basis of $\mathbb{R}^n$.

4. (10 points) Suppose V is a complex vector space of dimension n over $\mathbb{C}$. Complete the following proof that V has dimension $2n$ over $\mathbb{R}$.

Proof. Let $v_1, \ldots, v_n \in V$ be a basis for V over $\mathbb{C}$. We claim that

$$\underline{\hspace{6cm}}$$

form a basis for V over $\mathbb{R}$. First, suppose that

$$a_1 v_1 + b_1(iv_1) + \cdots + a_n v_n + b_n(iv_n) = 0$$

for some real scalars $a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$. This equation is the same as

$$(a_1 + ib_1)v_1 + \underline{\hspace{4cm}} = 0.$$

Since $v_1, \ldots, v_n$ are linearly independent over $\mathbb{C}$ (because they form a basis for V over $\mathbb{C}$), all coefficients above must be zero:

$$\underline{\hspace{6cm}}.$$

But a complex number is zero if and only if both its real and imaginary parts are zero, so we conclude that

$$\underline{\hspace{6cm}},$$

and hence $\underline{\hspace{5cm}}$ are linearly independent over $\mathbb{R}$.

Let $w \in V$. Since $v_1, \ldots, v_n$ span V over $\mathbb{C}$, there are complex scalars $a_j + ib_j \in \mathbb{C}$ (with $a_j, b_j \in \mathbb{R}$) satisfying

$$w = \underline{\hspace{7cm}}.$$

But this can be written as

$$w = \underline{\hspace{8cm}},$$

which expresses w as a linear combination of $\underline{\hspace{5cm}}$ over $\mathbb{R}$. Hence these vectors span V over $\mathbb{R}$, so they form a basis for V over $\mathbb{R}$. There are $2n$ vectors in this basis, so V has dimension $2n$ over $\mathbb{R}$. $\square$

5. (10 points) Let W be the set of all polynomials $p(x)$ in $P_3(\mathbb{R})$ such that $p''(x)+p'(x)+p(x) = 0$.
(a) Show that W is a subspace of $P_3(\mathbb{R})$.
(b) Find a basis for W and hence determine the dimension of W.