Math 291-1: Midterm 2
Northwestern University, Fall 2017

Name: ____

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) There is no 2×2 matrix A such that $A^2 \neq I$ but $A^4 = I$.

(b) There is no vector space over $\mathbb{C}$ which has dimension 5 over $\mathbb{R}$.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose A is an $n \times n$ matrix and that $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ are linearly independent vectors such that

$$A\mathbf{v}_1 = \mathbf{v}_2,\ A\mathbf{v}_2 = \mathbf{v}_3,\ \ldots,\ A\mathbf{v}_{n-1} = \mathbf{v}_n,\ \text{and } A\mathbf{v}_n = \mathbf{v}_1.$$

To be clear, A has the effect of sending each of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to the next vector in the list, except that $\mathbf{v}_n$ is sent to $\mathbf{v}_1$. Show that $A^n = I$.

3. (10 points) Suppose A and B are $n \times n$ matrices such that $AB = I_n$. Show that A and B are each invertible. (You cannot use the fact that if $AB = I_n$ for square matrices, then $BA = I_n$ automatically since that fact relies on the claim given here. You also cannot use the fact that if AB is invertible, then A and B are each invertible, since this also relies on the claim given here.) Hint: Show that B is invertible first, using some aspect of the Amazingly Awesome Theorem.

4. (10 points) Suppose V is a vector space over $\mathbb{K}$ and that U is a (linear) subspace of V. Suppose $b \in V$ is not in U, and define $b + U$ to be the set of all vectors in V obtained by adding V to elements of U:

$$b + U = \{b + u \in V \mid u \in U\}.$$

Let $w_1, \ldots, w_k \in b + U$ and $c_1, \ldots, c_k \in \mathbb{K}$. Show that $c_1 w_1 + \cdots + c_k w_k \in b + U$ if and only if $c_1 + \cdots + c_k = 1$. (You cannot take it for granted that $b + U$ is an affine subspace of V, since this fact is a consequence of this problem.)

Be careful: the forward direction, namely that if $c_1 w_1 + \cdots + c_k w_k \in b + U$ then $c_1 + \cdots + c_k = 1$, is not as obvious as it seems and requires some real thought.

5. (10 points) Let U be the subset of $M_2(\mathbb{C})$ consisting of all 2×2 complex matrices which equal their own transpose:

$$U := \{A \in M_2(\mathbb{C}) \mid A^T = A\}.$$

Show that U is a subspace of $M_2(\mathbb{C})$ over $\mathbb{R}$, and find a basis for U over $\mathbb{R}$. You can take it for granted that $(A+B)^T = A^T + B^T$ and $(cA)^T = cA^T$, where c is a scalar. You do NOT have to justify the fact that your claimed basis is actually a basis.