Math 291-1: Midterm 2 Solutions
Northwestern University, Fall 2015

1. Determine, with justification, whether each of the following is true or false.
(a) There exists a 2×2 non-identity matrix A such that $A^5 = I$.
(b) The space of 3×3 upper-triangular complex matrices has a 7-dimensional complex subspace.

Solutions. (a) This is true. Let A be the matrix describing rotation by $2\pi/5$, so concretely

$$A = \begin{pmatrix} \cos(2\pi/5) & -\sin(2\pi/5) \\ \sin(2\pi/5) & \cos(2\pi/5) \end{pmatrix}.$$

This is not the identity but A^5 describes rotation by 2π, which is the identity.
(b) This is false. A 3×3 upper-triangular complete matrix is of the form

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \text{ where } a, b, c, d, e, f \in \mathbb{C}.$$

This has 6 independent parameters, so the space of 3×3 upper-triangular complex matrices is 6-dimensional over $\mathbb{C}$ and hence can't have a subspace of larger dimension than 6. ☐

2. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$T\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \text{ and } T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and that $S : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$S\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } S\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If A denotes the standard matrix of the composition ST, compute A^2 explicitly and explain why there is not enough information to determine the standard matrix of TS.

Solution. We have

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}T\begin{pmatrix} 0 \\ -2 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$(ST)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = S\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} = -S\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

which gives the second column of A. Next,

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = T\left(\frac{1}{2}\begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \frac{1}{2}T\begin{pmatrix} 0 \\ -2 \end{pmatrix} + T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix},$$

so

$$(ST)\begin{pmatrix}1\\0\end{pmatrix} = S\begin{pmatrix}2\\0\\1\end{pmatrix} = S\left(\begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}\right) = S\begin{pmatrix}2\\0\\0\end{pmatrix} + S\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix},$$

which gives the first column of A. Hence

$$A = \begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}, \text{ so } A^2 = \begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & -1\end{pmatrix}.$$

To compute the standard matrix of TS we would have to know the value of

$$(TS)\begin{pmatrix}0\\1\\0\end{pmatrix},$$

which cannot be determined from the given information since $\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$ is not a linear combination of $\left(\begin{smallmatrix}2\\0\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$. A more high-brow way of saying this is that TS is uniquely determined by 3 pieces of linearly independent data, but we are only given two. $\square$

3. Suppose $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are linearly independent and that A is a 2×2 matrix such that $A\mathbf{v}_1 = \mathbf{v}_2$ and $A\mathbf{v}_2 = \mathbf{v}_1$. Show that the columns of A span $\mathbb{R}^2$.

Bonus (1 extra point): If $\mathbf{v}_1$ and $\mathbf{v}_2$ have the same length, show that A describes a reflection.

Proof. The key point is to show that A is invertible. There are multiple ways of doing this—here are a few.

First, from the given information we have

$$A\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix} = \begin{pmatrix}\mathbf{v}_2 & \mathbf{v}_1\end{pmatrix}$$

where the two matrices here apart from A have $\mathbf{v}_1, \mathbf{v}_2$, in some order, as their columns. Since $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, $\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}$ is invertible so

$$A = \begin{pmatrix}\mathbf{v}_2 & \mathbf{v}_1\end{pmatrix}\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}^{-1}.$$

Since A is thus the product of two invertible matrices, it is invertible itself and so its columns span $\mathbb{R}^2$.

Second, suppose that $\mathbf{x} \in \mathbb{R}^2$ satisfies $A\mathbf{x} = \mathbf{0}$. Since $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent in a 2-dimensional space, they must span $\mathbb{R}^2$ so

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some $c_1, c_2 \in \mathbb{R}$. Then

$$\mathbf{0} = A\mathbf{x} = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1\mathbf{v}_2 + c_2\mathbf{v}_1,$$

which since $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent implies $c_1 = c_2 = 0$. Then $\mathbf{x} = 0\mathbf{v}_1 + 0\mathbf{v}_2 = \mathbf{0}$, so the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Thus A is invertible, so its columns span $\mathbb{R}^2$.

Third, let $\mathbf{b} \in \mathbb{R}^2$. As above, $\mathbf{v}_1, \mathbf{v}_2$ span $\mathbb{R}^2$ so

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some $c_1, c_2 \in \mathbb{R}$. Then

$$A(c_2\mathbf{v}_1 + c_1\mathbf{v}_2) = c_2A\mathbf{v}_1 + c_1A\mathbf{v}_2 = c_2\mathbf{v}_2 + c_1\mathbf{v}_1 = \mathbf{b},$$

which shows that $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b} \in \mathbb{R}^2$. Hence A is invertible, so its columns span $\mathbb{R}^2$.

Fourth, let $\mathbf{x} \in \mathbb{R}^2$. As above, $\mathbf{v}_1, \mathbf{v}_2$ span $\mathbb{R}^2$ so

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some $c_1, c_2 \in \mathbb{R}$. Then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2) = A(c_1\mathbf{v}_2 + c_2\mathbf{v}_1) = c_1A\mathbf{v}_2 + c_2A\mathbf{v}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}.$$

Since $A^2\mathbf{x} = \mathbf{x}$ for any $\mathbf{x}$, $A^2 = I$, so A is invertible and equals its own inverse. Thus the columns of A span $\mathbb{R}^2$.

Bonus: Let L be the line which bisects the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, and let R denote the transformation which reflects vectors across this line. Then $R\mathbf{v}_1 = \mathbf{v}_2 = A\mathbf{v}_1$ and $R\mathbf{v}_2 = \mathbf{v}_1 = A\mathbf{v}_2$. (Note this wouldn't be true if $\mathbf{v}_1$ and $\mathbf{v}_2$ did not have the same length.) Since $\mathbf{v}_1, \mathbf{v}_2$ span $\mathbb{R}^2$, we can write an arbitrary $\mathbf{x} \in \mathbb{R}^2$ as $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, so

$$A\mathbf{x} = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1R\mathbf{v}_1 + c_2R\mathbf{v}_2 = R\mathbf{x}.$$

Since $A\mathbf{x} = R\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$, $A = R$ so A is a reflection. □

4. Let A be an $n \times n$ matrix and suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfy

$$A\mathbf{x} = \mathbf{x} \quad \text{and} \quad A\mathbf{y} = 3\mathbf{y}.$$

Also, let U be a subspace of $\mathbb{R}^n$ with the property that if $\mathbf{u} \in U$, then $A\mathbf{u} \in U$ as well. If $\mathbf{x}+\mathbf{y} \in U$, show that $\mathbf{x} \in U$ and $\mathbf{y} \in U$.

Proof. By the property U has, since $\mathbf{x} + \mathbf{y} \in U$, we also have

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{x} + 3\mathbf{y} \in U.$$

Since $\mathbf{x} + \mathbf{y}, \mathbf{x} + 3\mathbf{y} \in U$,

$$(\mathbf{x} + \mathbf{y}) - (\mathbf{x} + 3\mathbf{y}) = -3\mathbf{y} \in U$$

since U is closed under differences (to be clear, $-(\mathbf{x} + 3\mathbf{y})$ is in U since U is closed under scalar multiplication, and then $(\mathbf{x} + \mathbf{y}) + [-(\mathbf{x} + 3\mathbf{y})] \in U$ since U is closed under addition. Since U is closed under scalar multiplication,

$$-\frac{1}{3}(-3\mathbf{y}) = \mathbf{y} \in U.$$

Since $\mathbf{x} + \mathbf{y} \in U$ and $\mathbf{y} \in U$, we get

$$(\mathbf{x} + \mathbf{y}) - \mathbf{y} = \mathbf{x} \in U$$

as well, so $\mathbf{x}, \mathbf{y}$ are both in U. □

5. Let W be the subspace of $P_3(\mathbb{C})$ consisting of the polynomials $p(x) \in P_3(\mathbb{C})$ which satisfy

$$p(x) + p(-x) = 0.$$

Find a basis for W considered as a vector space over $\mathbb{R}$.

Proof. If $p(x) = c_0 + c_1x + c_2x^2 + c_3x^3$ with $c_0, c_1, c_2, c_3 \in \mathbb{C}$, to be in W this must satisfy

$$p(x) + p(-x) = (c_0 + c_1x + c_2x^2 + c_3x^3) + (c_0 - c_1x + c_2x^2 - c_3x^3) = 2c_0 + 2c_2x^2 = 0.$$

This forces $2c_0 = 2c_2 = 0$ since 1 and x^2 are linearly independent, so $c_0 = c_2 = 0$. Thus an element of W is of the form

$$p(x) = c_1x + c_3x^3 \text{ where } c_1, c_3 \in \mathbb{C}.$$

Writing $c_1 = a_1 + ib_1$ and $c_3 = a_3 + ib_3$ where $a_1, b_1, a_3, b_3 \in \mathbb{R}$, this becomes

$$p(x) = (a_1 + ib_1)x + (a_3 + ib_3)x^3 = a_1x + b_1(ix) + a_3x^3 + b_3(ix^3),$$

which shows that x, ix, x^3, ix^3 span W over $\mathbb{R}$.

To check that these are linearly independent over $\mathbb{R}$, suppose

$$a_1x + b_1(ix) + a_3x^3 + b_3(ix^3) = 0$$

for some $a_1, b_1, a_3, b_3 \in \mathbb{R}$. Then

$$(a_1 + ib_1)x + (a_3 + ib_3)x^3 = 0,$$

so $a_1 + ib_1 = 0$ and $a_3 + ib_3 = 0$ since x, x^3 are linearly independent. Hence $a_1 = b_1 = 0$ and $a_3 = b_3 = 0$, so x, ix, x^3, ix^3 are independent over $\mathbb{R}$, and hence form a basis for W over $\mathbb{R}$. $\square$