Math 291-1: Midterm 2 Solutions
Northwestern University, Fall 2016

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample. (A counterexample is a specific example in which the given claim is indeed false.)

(a) If $A \in M_2(\mathbb{R})$ describes reflection across a line passing through the origin, then A is invertible.

(b) The space $M_4(\mathbb{R})$ does not have a 5-dimensional subspace.

Solution. (a) This is true. A reflection satisfies $A^2\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$, since reflecting a vector, then reflecting the result will ways give back the original vector. Hence $A^2 = I$, so $AA = I$ and A is invertible since there is a matrix (namely A itself) such that multiplying by A on either side gives the identity. (In other words, a reflection is its own inverse.)

(b) This is false. For instance the set of all matrices of the form

$$\begin{bmatrix} a & b & c & d \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a 5-dimensional subspace of $M_4(\mathbb{R})$. Indeed, this set is just the span of

$$E_{11}, E_{12}, E_{13}, E_{14}, E_{21},$$

where E_{ij} is just the matrix which as a 1 in row i, column j and zeroes everywhere else. This span is a subspace of $M_4(\mathbb{R})$, and since $E_{11}, E_{12}, E_{13}, E_{14}, E_{21}$ form a basis for it, this subspace is 5-dimensional. $\square$

2. On the board (or in a separate file) is a proof that if A is a 2×2 matrix and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are linearly independent vectors such that

$$A\mathbf{v}_1 = \mathbf{0} \text{ and } A\mathbf{v}_2 \in \operatorname{span}(\mathbf{v}_1),$$

then $A^2 = 0$. Using this as a guide, prove that if A is an $n \times n$ matrix and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ are linearly independent vectors such that

$$A\mathbf{v}_1 = \mathbf{0} \text{ and } A\mathbf{v}_k \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) \text{ for } k = 2, \ldots, n,$$

then $A^n = 0$.

Proof. We first claim that $A^\ell \mathbf{v}_\ell = \mathbf{0}$ for each $\ell = 1, \ldots, n$. Indeed, we're given that $A\mathbf{v}_1 = \mathbf{0}$. Since $A\mathbf{v}_2 \in \operatorname{span}(\mathbf{v}_1)$, we have $A\mathbf{v}_2 = b\mathbf{v}_1$ for some $b \in \mathbb{R}$. Then

$$A^2\mathbf{v}_2 = A(A\mathbf{v}_2) = A(b\mathbf{v}_1) = bA\mathbf{v}_1 = b\mathbf{0} = \mathbf{0}.$$

Now, $A\mathbf{v}_3 \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$, so

$$A\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \text{ for some } c_1, c_2 \in \mathbb{R}.$$

Thus

$$A^2\mathbf{v}_3 = A(A\mathbf{v}_3) = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 = c_2 b\mathbf{v}_1,$$

and then

$$A^3\mathbf{v}_3 = A(A^2\mathbf{v}_3) = A(c_2 b\mathbf{v}_1) = c_2 bA\mathbf{v}_1 = \mathbf{0}.$$

Notice the pattern in the computations above: $A\mathbf{v}_3$ depended on $\mathbf{v}_1$ and $\mathbf{v}_2$, and after multiplying by A we see that one term is "killed off" so that $A\mathbf{v}_3$ only depends on $\mathbf{v}_1$, and then multiplying by A once more kills this final term off as well.

The same reasoning will show that $A\mathbf{v}_4$ depends on $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, $A^2\mathbf{v}_4$ only depends on $\mathbf{v}_1, \mathbf{v}_2$, $A^3\mathbf{v}_4$ only depends on $\mathbf{v}_1$, and $A^4\mathbf{v}_4$ is zero, and that the pattern continues for other $\mathbf{v}_k$. To phrase this all more clearly, consider the cases worked out above as base cases and suppose we have shown that $A^i\mathbf{v}_i = \mathbf{0}$ for all i up to some k. (You did not have to write this out as formally in your own solution; noting the pattern above would have been enough.) Note that this implies

$$A^k\mathbf{v}_i = \mathbf{0} \text{ for all } i \leq k$$

as well since once $A^i\mathbf{v}_i$ is zero, multiplying by more powers of A will still give zero. Then since $A\mathbf{v}_{k+1} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$, we have

$$A\mathbf{v}_{k+1} = d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k \text{ for some } d_1, \ldots, d_k \in \mathbb{R}.$$

This gives:

$$A^{k+1}\mathbf{v}_{k+1} = A^k(A\mathbf{v}_{k+1}) = A^k(d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k) = d_1A^k\mathbf{v}_1 + \cdots + d_kA^k\mathbf{v}_k = \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}.$$

Thus knowing that $A^i\mathbf{v}_i = \mathbf{0}$ for all $i \leq k$ implies that $A^{k+1}\mathbf{v}_{k+1} = \mathbf{0}$ as well, so we conclude that

$$A^\ell\mathbf{v}_\ell = \mathbf{0} \text{ for all } \ell.$$

Since multiplying $\mathbf{0}$ by more powers of A still gives $\mathbf{0}$, this implies that

$$A^n\mathbf{v}_k = \mathbf{0} \text{ for all } 1 \leq k \leq n.$$

Now, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent vectors in $\mathbb{R}^n$, so they automatically span $\mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$. Then

$$\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \text{ for some } a_1, \ldots, a_n \in \mathbb{R}.$$

Hence

$$A^n\mathbf{x} = a_1A^n\mathbf{v}_1 + \cdots + a_nA^n\mathbf{v}_n = \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0},$$

so $A^n\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$. Thus $A^n = 0$ as claimed. (The point is that one you know $A^n = 0$ on a basis of $\mathbb{R}^n$, it must be zero on all of $\mathbb{R}^n$.) □

3. Suppose A is an $n \times n$ matrix and that $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ form a basis of $\mathbb{R}^n$. Show that A is invertible if and only if $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ form a basis of $\mathbb{R}^n$.

Proof 1. Let $\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}$ be the $n \times n$ matrix with the given basis vectors as its columns. This matrix is invertible by the Amazingly Awesome Theorem since its columns form a basis of $\mathbb{R}^n$. Then

$$A\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}A\mathbf{v}_1 & \cdots & A\mathbf{v}_n\end{bmatrix},$$

where the matrix on the right is the one with $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ as columns. If A is invertible, then the product on the left is invertible since it is a product of invertible matrices, so

$$\begin{bmatrix}A\mathbf{v}_1 & \cdots & A\mathbf{v}_n\end{bmatrix}$$

is invertible and hence $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ form a basis of $\mathbb{R}^n$ by the Amazingly Awesome Theorem.

If instead $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ form a basis for $\mathbb{R}^n$, then the matrix on the right is invertible. Multiplying both sides on the right by the inverse of $\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}$ gives

$$A = \begin{bmatrix}A\mathbf{v}_1 & \cdots & A\mathbf{v}_n\end{bmatrix}\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}^{-1},$$

which expresses A as a product of invertible matrices. Hence A is invertible as claimed. $\square$

Proof 2. Suppose A is invertible and suppose

$$c_1 A\mathbf{v}_1 + \cdots + c_n A\mathbf{v}_n = \mathbf{0}.$$

Multiplying through by A^{-1} gives

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = A^{-1}\mathbf{0} = \mathbf{0}.$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, we must have $c_1 = \cdots = c_n = 0$, so we conclude that $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ are linearly independent. Since these are n linearly independent vectors in an n-dimensional space, they automatically form a basis.

Conversely suppose $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ form a basis of $\mathbb{R}^n$ and suppose $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$, we have

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \text{ for some } c_1, \ldots, c_n \in \mathbb{R}.$$

Then

$$\mathbf{0} = A\mathbf{x} = c_1 A\mathbf{v}_1 + \cdots + c_n A\mathbf{v}_n.$$

Since $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ are linearly independent, $c_1, \ldots, c_n$ are all zero, so

$$\mathbf{x} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

Thus the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, so A is invertible by the Amazingly Awesome Theorem. (There are of course other possible proofs, using other aspects of this theorem.) $\square$

4. Suppose V is a complex vector space of dimension n over $\mathbb{C}$. Complete the following proof that V has dimension $2n$ over $\mathbb{R}$.

Proof. Let $v_1, \ldots, v_n \in V$ be a basis for V over $\mathbb{C}$. We claim that

$$\underline{v_1,\ iv_1,\ v_2,\ iv_2, \ldots, v_n,\ iv_n}$$

form a basis for V over $\mathbb{R}$. First, suppose that

$$a_1v_1 + b_1(iv_1) + \cdots + a_nv_n + b_n(iv_n) = 0$$

for some real scalars $a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$. This equation is the same as

$$(a_1 + ib_1)v_1 + \underline{\cdots + (a_n + ib_n)v_n} = 0.$$

Since $v_1, \ldots, v_n$ are linearly independent over $\mathbb{C}$ (because they form a basis for V over $\mathbb{C}$), all coefficients above must be zero:

$$\underline{a_1 + ib_1 = 0, \ldots, a_n + ib_n = 0}.$$

But a complex number is zero if and only if both its real and imaginary parts are zero, so we conclude that

$$\underline{a_1 = 0 = b_1, \ldots, a_n = 0 = b_n},$$

and hence $\underline{v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n}$ are linearly independent over $\mathbb{R}$.

Let $w \in V$. Since $v_1, \ldots, v_n$ span V over $\mathbb{C}$, there are complex scalars $a_j + ib_j \in \mathbb{C}$ (with $a_j, b_j \in \mathbb{R}$) satisfying

$$w = \underline{(a_1 + ib_1)v_1 + \cdots + (a_n + ib_n)v_n}.$$

But this can be written as

$$w = \underline{a_1 v_1 + b_1(iv_1) + \cdots + a_n v_n + b_n(iv_n)},$$

which expresses w as a linear combination of $\underline{v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n}$ over $\mathbb{R}$. Hence these vectors span V over $\mathbb{R}$, so they form a basis for V over $\mathbb{R}$. There are $2n$ vectors in this basis, so V has dimension $2n$ over $\mathbb{R}$. $\square$

5. Let W be the set of all polynomials $p(x)$ in $P_3(\mathbb{R})$ such that $p''(x) + p'(x) + p(x) = 0$.

(a) Show that W is a subspace of $P_3(\mathbb{R})$.

(b) Find a basis for W and hence determine the dimension of W.

Solution. (a) First, the constant zero polynomial 0 satisfies

$$0'' + 0' + 0 = 0 + 0 + 0 = 0,$$

so $0 \in W$. If $p(x), q(x) \in W$, then

$$\begin{aligned} (p(x) + q(x))'' + (p(x) + q(x))' + (p(x) + q(x)) &= p''(x) + q''(x) + p'(x) + q'(x) + p(x) + q(x) \\ &= (p''(x) + p'(x) + p(x)) + (q''(x) + q'(x) + q(x)) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

where in the third line we use the fact that $p(x)$ and $q(x)$ are both in W in order to say that the previous expressions were zero. Hence W is closed under addition. Finally, with $p(x) \in W$ and $c \in \mathbb{R}$, we have

$$(cp(x))'' + (cp(x))' + (cp(x)) = cp''(x) + cp'(x) + cp(x) = c(p''(x) + p'(x) + p(x)) = c0 = 0,$$

so $cp(x) \in W$ and W is closed under scalar multiplication. Thus W is a subspace of $P_3(\mathbb{R})$.

(b) We first determine explicitly what an element of W looks like. Suppose

$$p(x) = a + bx + cx^2 + dx^3 \in W.$$

Then $p''(x) + p'(x) + p(x) = 0$, so

$$(2c + 6dx) + (b + 2cx + 3dx^2) + (a + bx + cx^2 + dx^3) = 0.$$

Rearranging gives

$$(2c + b + a) + (6d + 2c + b)x + (3d + c)x^2 + dx^3 = 0.$$

In order for a polynomial to be zero requires that the coefficients of x^k each be zero, so we get that

$$\begin{aligned} a + b + 2c \quad\quad &= 0 \\ b + 2c + 6d &= 0 \\ c + 3d &= 0 \\ d &= 0. \end{aligned}$$

Solving this system gives $a = b = c = d = 0$, so $p(x) = 0$. Hence only thing in W is the zero polynomial, so $W = \{0\}$ and is thus zero dimensional. By convention, a basis for W is the empty set $\emptyset$ (with nothing in it), but no points were deducted for missing this subtle point. $\square$