Math 291-3: Midterm 2 Solutions
Northwestern University, Spring 2016

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) Any C^1 closed 1-form on the region obtained by removing $(1,1)$ from $\mathbb{R}^2$ is exact.

(b) If $f : \mathbb{R}^3 \to \mathbb{R}$ is C^2, then the 2-form

$$\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\right) dz \wedge dx + \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

is closed on $\mathbb{R}^3$.

Solution. (a) This is false. The 1-form

$$\omega = \frac{-(y-1)\,dx + (x-1)\,dy}{(x-1)^2 + (y-1)^2}$$

is closed on the given region, as a direct computation shows. However, ω is not exact since its integral over a circle centered at $(1,1)$ will not be zero, as can be seen using the parametric equations $x = 1 + \cos t,\ y = 1 + \sin t$.

(b) This is true. The exterior derivative of the given 2-form is:

$$\begin{aligned}
&d\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\right) \wedge dy \wedge dz + d\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\right) \wedge dz \wedge dx + d\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) \wedge dx \wedge dy \\
&= \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial z}\right) dx \wedge dy \wedge dz + \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial x}\right) dy \wedge dz \wedge dx + \\
&\quad \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial z \partial y}\right) dz \wedge dx \wedge dy \\
&= \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial z \partial y}\right) dx \wedge dy \wedge dz.
\end{aligned}$$

Since f is C^2, all terms in this coefficient cancel out, so the given 2-form is closed. □

2. Suppose S is a smooth C^1 surface with parametrization

$$\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v)),\ (u,v) \in E$$

where E is a subset of $\mathbb{R}^2$, and let $\mathbf{c}(t) = (u(t), v(t)),\ a \le t \le b$ be a parametrization of a smooth C^1 curve in E. The composition $\mathbf{X} \circ \mathbf{c} : [a,b] \to \mathbb{R}^3$ then describes a smooth C^1 curve on S. Show that for any $t \in [a,b]$,

$$(\mathbf{X} \circ \mathbf{c})'(t) \cdot (\mathbf{X}_u \times \mathbf{X}_v)(u(t), v(t)) = 0.$$

Hint: Show that $(\mathbf{X} \circ \mathbf{c})'(t)$ is a linear combination of $\mathbf{X}_u(u(t), v(t))$ and $\mathbf{X}_v(u(t), v(t))$. (The point is that $(\mathbf{X} \circ \mathbf{c})'(t)$ gives a vector tangent to S at the point $\mathbf{X}(u(t), v(t))$, so this verifies that $(\mathbf{X}_u \times \mathbf{X}_v)(u(t), v(t))$ is orthogonal to every vector which is tangent to S at $\mathbf{X}(u(t), v(t))$, which is why $\mathbf{X}_u \times \mathbf{X}_v$ indeed gives vectors normal to S.)

Proof. We have

$$(\mathbf{X}\circ\mathbf{c})(t) = (x(u(t),v(t)), y(u(t),v(t)), z(u(t),v(t))).$$

The chain rule gives

$$\begin{aligned}(\mathbf{X}\circ\mathbf{c})'(t) &= \left(\frac{\partial x}{\partial u}u'(t) + \frac{\partial x}{\partial v}v'(t), \frac{\partial y}{\partial u}u'(t) + \frac{\partial y}{\partial z}v'(t), \frac{\partial z}{\partial u}u'(t) + \frac{\partial z}{\partial v}v'(t)\right)\\ &= u'(t)\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) + v'(t)\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\\ &= u'(t)\mathbf{X}_u + v'(t)\mathbf{X}_v.\end{aligned}$$

Thus

$$(\mathbf{X}\circ\mathbf{c})'(t)\cdot(\mathbf{X}_u\times\mathbf{X}_v) = u'(t)\mathbf{X}_u\cdot(\mathbf{X}_u\times\mathbf{X}_v) + v'(t)\mathbf{X}_v\cdot(\mathbf{X}_u\times\mathbf{X}_v).$$

The cross product $\mathbf{X}_u\times\mathbf{X}_v$ is orthogonal to both $\mathbf{X}_u$ and $\mathbf{X}_v$, so both dot products above are zero and hence $(\mathbf{X}\circ\mathbf{c})'(t)\cdot(\mathbf{X}_u\times\mathbf{X}_v)(u(t),v(t)) = 0$ as claimed. □

3. Suppose $\mathbf{F},\mathbf{G}$ are two C^1 vector fields on $\mathbb{R}^3$ which are orthogonal at every point, meaning

$$\mathbf{F}(\mathbf{p})\cdot\mathbf{G}(\mathbf{p}) = 0 \text{ for all } \mathbf{p}\in\mathbb{R}^3.$$

If C is a curve lying on a flow line of $\mathbf{G}$, determine the value of $\int_C \mathbf{F}\cdot d\mathbf{s}$.

Proof. Let $\mathbf{x}(t),\ a \le tleb$ be a parametrization of C. Then

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt.$$

Since C lies on a flow line of $\mathbf{G}$, $\mathbf{x}'(t) = \mathbf{G}(\mathbf{x}(t))$, so the integral above is

$$\int_a^b \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{G}(\mathbf{x}(t))\,dt.$$

Since $\mathbf{F}$ and $\mathbf{G}$ are orthogonal at every point, the integrand in the integral above is zero, and hence the line integral in question has the value zero. □

4. Compute

$$\int_C (2xy+yz)\,dx + (x^2+xz-z-2yz^3)\,dy + (y+xy-3y^2z^2)\,dz$$

where C is the intersection of the plane $y = z$ with the cylinder $x^2+z^2 = 1$, oriented counterclockwise when viewed from the positive z-axis.

Solution. We can write the given 1-form as

$$d(x^2y+xyz-y^2z^2) - z\,dy + y\,dz.$$

Thus the given integral is

$$\int_C d(x^2y+xyz-y^2z^2) + \int_C -z\,dy + y\,dz.$$

Since C is closed, the first integral above is zero. To compute the second, we parametrize the given curve as

$$\mathbf{x}(t) = (\cos t, \sin t, \sin t),\ 0 \le t \le 2\pi.$$

Then

$$\int_C -z\,dy + y\,dz = \int_0^{2\pi} (-\sin t\cos t + \sin t\cos t)\,dt = \int_0^{2\pi} 0\,dt = 0.$$

Thus the integral in question has the value zero. $\square$

5. Suppose C is a smooth, C^1 curve in $\mathbb{R}^2$ which starts at $(4,0)$, ends at $(-3,0)$, and otherwise lies fully above the x-axis. Show that

$$\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \pi.$$

Solution. Pick $r > 0$ small enough to guarantee that the circle of radius r centered at $(0,0)$ lies fully below the curve C. Let C_1 be the curve consisting of the line segment from $(-3,0)$ to $(-r,0)$, followed by the top half of the circle of radius r centered at $(0,0)$, followed by the line segment from $(r,0)$ to $(4,0)$. Then $C+C_1$ is the boundary of the region D lying between these two curves. Since the given 1-form is C^1 throughout D (note that D does not include the origin), Green's Theorem applies to give

$$\int_{C+C_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = \iint_D d\left(\frac{-y\,dx + x\,dy}{x^2+y^2}\right) = \iint_D 0\,dA = 0.$$

Thus

$$\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = -\int_{C_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_{-C_1} \frac{-y\,dx + x\,dy}{x^2+y^2}.$$

Now, this final integral beaks up into

$$\int_{\text{first line segment}} \frac{-y\,dx + x\,dy}{x^2+y^2} + \int_{-\text{semicircle}} \frac{-y\,dx + x\,dy}{x^2+y^2} + \int_{\text{final line segment}} \frac{-y\,dx + x\,dy}{x^2+y^2}.$$

Over the first line segment, $y = 0$ is constant, so $dy = 0$ on this segment and hence the first integral is zero. Over the final segment, again $y = 0$ is constant, so $dy = 0$ and hence the third integral above is zero. To compute the middle integral, we parametrize the semicircle using

$$\mathbf{x}(t) = (r\cos t, r\sin t),\ 0 \le t \le \pi.$$

Then

$$\int_{-\text{semicircle}} \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{\pi} \left(\frac{r^2\sin^2 t + r^2\cos^2 t}{r^2}\right) dt = \int_0^{\pi} dt = \pi.$$

Hence

$$\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \pi$$

as claimed. $\square$