Math 291-3: Midterm 2 Solutions
Northwestern University, Spring 2017

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If $\mathbf{F}$ is C^1 and satisfies $\operatorname{div}\mathbf{F} = x$, then there does not exist a C^2 field $\mathbf{G}$ such that $\operatorname{curl}\mathbf{G} = \mathbf{F}$.

(b) If C is a curve and $\int_C \mathbf{F}\cdot d\mathbf{s} = 0$, then $\mathbf{F}$ is conservative.

Solution. (a) This is true. If there did exist such a $\mathbf{G}$, we would have $\operatorname{div}\mathbf{F} = \operatorname{div}(\operatorname{curl}\mathbf{G}) = 0$.

(b) This is false. Take $\mathbf{F} = xy\mathbf{i}$. Then $\mathbf{F}$ is not conservative (for instance, it's curl is nonzero), but integrating $\mathbf{F}$ over a vertical line segment would give zero since $\mathbf{F}$ has no $\mathbf{j}$-component. □

2. Recall that the surface area of a smooth C^1 surface with parametrization $\mathbf{X}(u, v)$ where $(u, v) \in D$ is given by

$$\iint_D \|\mathbf{X}_u \times \mathbf{X}_v\|\, du\, dv.$$

Compute the surface area of the portion of the cone $z = \sqrt{x^2 + y^2}$ lying below $z = 4$.

Solution. We parametrize the portion of the cone we want using:

$$\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, r) \text{ for } 0 \le r \le 4, 0 \le \theta \le 2\pi.$$

We have

$$\mathbf{X}_r = (\cos\theta, \sin\theta, 1) \text{ and } \mathbf{X}_\theta = (-r\sin\theta, r\cos\theta, 0),$$

so

$$\mathbf{X}_r \times \mathbf{X}_\theta = (-r\cos\theta, -r\sin\theta, r).$$

The surface area is thus given by:

$$\begin{aligned}\iint_D \|\mathbf{X}_r \times \mathbf{X}_\theta\|\, dr\, d\theta &= \int_0^{2\pi}\int_0^4 \sqrt{2r^2}\, dr\, d\theta \\ &= \sqrt{2}\int_0^{2\pi}\int_0^4 r\, dr\, d\theta \\ &= \sqrt{2}\int_0^{2\pi} 8\, d\theta \\ &= 16\pi\sqrt{2}.\end{aligned}$$

□

3. Suppose $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a C^2 vector field. Show that

$$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \nabla(\operatorname{div}\mathbf{F}) - \langle \operatorname{div}(\nabla P), \operatorname{div}(\nabla Q), \operatorname{div}(\nabla R)\rangle.$$

Start by computing the left-hand side.

Proof. First, we have:

$$\operatorname{curl}\mathbf{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y).$$

Thus:

$$\begin{aligned}
\operatorname{curl}(\operatorname{curl}\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ R_y - Q_z & P_z - R_x & Q_x - P_y \end{vmatrix} \\
&= (Q_{xy} - P_{yy} - P_{zz} + R_{xz}, R_{yz} - Q_{zz} - Q_{xx} + P_{yx}, P_{zx} - R_{xx} - R_{yy} + Q_{zy}).
\end{aligned}$$

Next we compute:

$$\nabla(\operatorname{div}\mathbf{F}) = \nabla(P_x + Q_y + R_z) = (P_{xx} + Q_{yx} + R_{zx}, P_{xy} + Q_{yy} + R_{zy}, P_{xz} + Q_{yz} + R_{xx}).$$

Finally,

$$\begin{aligned}
(\operatorname{div}\nabla P, \operatorname{div}\nabla Q, \operatorname{div}\nabla R) &= (\operatorname{div}(P_x, P_y, P_z), \operatorname{div}(Q_x, Q_y, Q_z), \operatorname{div}(R_x, R_y, R_z)) \\
&= (P_{xx} + P_{yy} + P_{zz}, Q_{xx} + Q_{yy} + Q_{zz}, R_{xx} + R_{yy} + R_{zz}).
\end{aligned}$$

Computing $\nabla(\operatorname{div}\mathbf{F}) - (\operatorname{div}\nabla P, \operatorname{div}\nabla Q, \operatorname{div}\nabla R)$ using these final two expressions and using the fact that each of P, Q, R is C^2, we get the expression computed above for $\operatorname{curl}\operatorname{curl}\mathbf{F}$ as claimed. $\square$

4. Suppose C is the curve consisting of the line segment from $(0,0)$ to $(1,2)$, followed by the line segment from $(1,2)$ to $(2,0)$. Compute the following line integral:

$$\int_C (2xye^{x^2y} + e^y)\,dx + x^2e^{x^2y}\,dy$$

Solution. Write the integrand as

$$(2xye^{x^2y}\,dx + x^2e^{x^2y}\,dy) + e^y\,dx.$$

The first term is the differential of the function $f(x,y) = e^{x^2y}$, so the Fundamental Theorem of Line Integrals gives:

$$\begin{aligned}
\int_C (2xye^{x^2y} + e^y)\,dx + x^2e^{x^2y}\,dy &= \int_C df + \int_C e^y\,dx \\
&= f(\text{end point}) - f(\text{start point}) + \int_C e^y\,dx. \\
&= f(2,0) - f(0,0) + \int_C e^y\,dx \\
&= e^0 - e^0 + \int_C e^y\,dx \\
&= \int_C e^y\,dx.
\end{aligned}$$

For this remaining integral we use parametric equations. The first line segment making up C has equation $y = 2x$, so we use

$$\mathbf{x}(t) = (t, 2t),\ 0 \le t \le 1.$$

Hence the line integral over this portion is

$$\int_0^1 e^{2t}\,dt = \frac{1}{2}(e^2 - 1).$$

The second line segment making up C has equation $y = 4 - 2x$, which we parametrize using

$$\mathbf{x}(t) = (t, 4 - 2t),\ 1 \leq t \leq 2.$$

Hence the line integral over this portion is

$$\int_1^2 e^{4-2t}\,dt = -\frac{1}{2}(1 - e^2) = \frac{1}{2}(e^2 - 1).$$

Adding these two thus gives our final value:

$$\int_C (2xye^{x^2y} + e^y)\,dx + x^2e^{x^2y}\,dy = e^2 - 1.$$

□

5. Suppose C is the ellipse $4x^2 + 9y^2 = 1$ oriented counterclockwise. Determine the value of the line integral

$$\int_C \frac{y\,dx - x\,dy}{x^2 + y^2},$$

justifying every step you take along the way. The only thing you may take for granted is that the exterior derivative of the 1-form in question is 0. Hint: Argue that you can replace C by a different curve.

Solution. Let C_1 be the unit circle centered at the origin, oriented clockwise. The combined curve $C + C_1$ is then the boundary of the region D lying outside the circle and within the ellipse. This curve has the correct orientation to be able to apply Green's Theorem, so Green's Theorem gives:

$$\oint_{\partial D} \frac{y\,dx - x\,dy}{x^2 + y^2} = \iint_D \left(\frac{\partial[-x/(x^2+y^2)]}{\partial x} - \frac{\partial[y/(x^2+y^2)]}{\partial y} \right) dA = \iint_D 0\,dA = 0,$$

where the 0 comes from direct computation or using fact which was stated we can take for granted. Now, since $\partial D = C + C_1$, this line integral can be split up into:

$$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2} + \oint_{C_1} \frac{y\,dx - x\,dy}{x^2 + y^2}$$

so since this sum is zero, we get

$$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2} = -\oint_{C_1} \frac{y\,dx - x\,dy}{x^2 + y^2} = \oint_{-C_1} \frac{y\,dx - x\,dy}{x^2 + y^2}$$

where $-C_1$ now denotes the unit circle with counterclockwise orientation. We parametrize this using

$$\mathbf{x}(t) = (\cos t, \sin t),\ 0 \leq t \leq 2\pi.$$

We thus have:

$$\oint_{-C_1} \frac{y\,dx - x\,dy}{x^2 + y^2} = \int_0^{2\pi} [\sin t(-\sin t) - \cos t(\cos t)]\,dt = \int_0^{2\pi} -1\,dt = -2\pi.$$

Thus

$$\int_C \frac{y\,dx - x\,dy}{x^2 + y^2} = -2\pi.$$

□