Math 291-2: Midterm 2 Solutions
Northwestern University, Winter 2016

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If a 2×2 matrix A only has 1 as an eigenvalue and $\binom{1}{1}, \binom{2}{2}$ are both eigenvectors corresponding to 1, then A is diagonalizable.

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation, then T is differentiable. (Recall that T being affine means T is of the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some 2×2 matrix A and $\mathbf{b} \in \mathbb{R}^2$.)

Solution. (a) This is false, with the point being that there is not enough information to say whether or not we can find two linearly independent eigenvectors. For example, the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

only has 1 as an eigenvalue and both $\binom{1}{1}$ and $\binom{2}{2}$ as eigenvectors, but is not diagonalizable.

To give a sense for how such an example can be found, note first that the characteristic polynomial of any such matrix must be $(\lambda - 1)^2 = \lambda^2 - 2\lambda + 1$. Thus if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is such a matrix, we must have $a + d = 2$ and $ad - bc = 1$. In addition, $\binom{1}{1}$ should be in $\ker(A - I)$, so

$$\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ must be } \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that a must be $1 - b$. Then it is a matter of coming up with a, b, c, d such that $a = 1 - b$, $a + d = 2$, $ad - bc = 1$, and checking that the example we come up with is indeed not diagonalizable. This isn't too hard of a guess and check.

(b) This is true. One reason is that if you write out $A\mathbf{x} + \mathbf{b}$, you get polynomial expressions, which are always differentiable. Alternatively, we have:

$$\frac{T(\mathbf{x} + \mathbf{h}) - T(\mathbf{x}) - A\mathbf{h}}{\|\mathbf{h}\|} = \frac{(A\mathbf{x} + A\mathbf{h} + \mathbf{b}) - (A\mathbf{x} + \mathbf{b}) - A\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0},$$

so the limit defining differentiability of T is 0. □

2. Suppose A is a symmetric $n \times n$ matrix. Show that there exists a symmetric $n \times n$ matrix B such that $B^5 = A$. Hint: Diagonalization.

Proof. Since A is symmetric, it is orthogonally diagonalizable so there exists an orthogonal $n \times n$ matrix Q and a diagonal $n \times n$ matrix D such that

$$A = QDQ^T.$$

Let $\lambda_1, \ldots, \lambda_n$ denote the diagonal entries of D, which are real since these are the eigenvalues of the symmetric matrix A. Any real number has a cube root, so set

$$B = Q \begin{pmatrix} \sqrt[5]{\lambda_1} & & \\ & \ddots & \\ & & \sqrt[5]{\lambda_n} \end{pmatrix} Q^T.$$

Then B is symmetric since it is orthogonally diagonalizable, and

$$B^5 = Q \begin{pmatrix} \sqrt[5]{\lambda_1} & & \\ & \ddots & \\ & & \sqrt[5]{\lambda_n} \end{pmatrix}^5 Q^T = Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^T = A$$

as required. (The overarching point here is that you can define all sorts of crazy operations on symmetric matrices: various roots, exponentials, sine and cosine, etc.) □

3. For $k \neq 0$, determine the point(s) on the surface

$$-x^2 + y^2 - z^2 + 4xz = k$$

which are closest to $(0,0,0)$. (The answer will depend on k.) You should justify that your answers are correct, but doing so based on the shape of the surface is good enough.

Solution. The left-hand side of the given equation defines a quadratic form with symmetric matrix

$$\begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

These has characteristic polynomial $(1-\lambda)(\lambda^2+2\lambda-3)$, and so has eigenvalues 1 and -3. Possible orthonormal eigenvectors for 1 are

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and a possible orthonormal eigenvector for -3 is

$$\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

After taking coordinates c_1, c_2, c_3 with respect to the basis of $\mathbb{R}^3$ consisting of these three eigenvectors, the equation for the surface becomes

$$c_1^2 + c_2^2 - 3c_3^2 = k.$$

For $k > 0$ this defines a hyperboloid of one sheet centered along the c_3-axis since for a fixed c_3

$$c_1^2 + c_2^2 = k + 3c_3^2$$

always describes a nonempty curve. The points closest to the origin on this hyperboloid are the ones making up the circle where the hyperboloid is the thinnest it can be, which occurs when $c_3 = 0$. Thus the points closest to the origin are those whose (c_1, c_2, c_3)-coordinates are

$$c_1, c_2 \text{ satisfying } c_1^2 + c_2^2 = k \text{ and } c_3 = 0,$$

or whose rectangular coordinates are

$$c_1 \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where $c_1^2 + c_2^2 = k$.

For $k < 0$ the given equation defined a hyperboloid of two sheets centered along the c_3-axis since for a fixed c_3

$$c_1^2 + c_2^2 = k + 3c_3^2$$

only describes a nonempty curve when $k + 3c_3^2 \geq 0$, so when $|c_3| \geq \sqrt{-k/3}$. (Note that $-k > 0$ since $k < 0$.) The points closest to the origin are those where the two sheets intersect the c_3-axis, and so occur when $c_1, c_2 = 0$. Thus in (c_1, c_2, c_3)-coordinates these points are

$$c_1 = c_2 = 0 \text{ and } c_3 = \pm\sqrt{\frac{-k}{3}},$$

and in rectangular coordinates they are

$$\pm\sqrt{\frac{-k}{3}} \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

Note that the $k = 0$ case was excluded in the setup, but nonetheless we can still give an answer here as well: when $k = 0$ we get a double cone which intersects the origin, so the point on this surface closest to the origin is the origin itself. □

4. Suppose that K and L are compact subsets of $\mathbb{R}^2$. Show that their union $K \cup L$ is compact as well. Hint: To show $K \cup L$ is closed, first show that $\partial(K \cup L) \subseteq \partial K \cup \partial L$.

Proof. Each of K and L is bounded since each is compact, so there exists $M > 0$ such that

$$\|\mathbf{x}\| \leq M \text{ for all } \mathbf{x} \in K$$

and there exists $N > 0$ such that

$$\|\mathbf{x}\| \leq N \text{ for all } \mathbf{x} \in L.$$

Thus

$$\|\mathbf{x}\| \leq \max\{M, N\} \text{ for all } \mathbf{x} \in K \cup L,$$

so $K \cup L$ is bounded.

Next we claim that $\partial(K \cup L) \subseteq \partial K \cup \partial L$. To see this, let $p \in \partial(K \cup L)$. If this p happens to be in ∂K then certainly it is in $\partial K \cup \partial L$, so we must show that if $p \notin \partial K$ then we must have $p \in \partial L$ instead. Thus suppose $p \notin \partial K$. Then there exists an open ball $B_r(p)$ around p which either contains no element of K or no element of K^c since p is not a boundary point of K. But since $p \in \partial(K \cup L)$, any ball around p, in particular $B_r(p)$, must contain an element outside of $K \cup L$. Thus $B_r(p)$ definitely contains something outside of K, so we conclude that $B_r(p)$ does not contain anything of K.

Since $p \in \partial(K \cup L)$, $B_r(p)$ contains an element of $K \cup L$, so this element must come from L since $B_r(p)$ contains no element of K; denote this element by $q \in L$. Then any ball larger than $B_r(p)$ still contains this same $q \in L$. Moreover, any ball smaller than $B_r(p)$ cannot contain an element of

K since then $B_r(p)$ would as well, so since $B_r(p)$ must contain an element of $K \cup L$ (because p is a boundary point of $K \cup L$), this element must always comes from L. All together this shows that any ball around p contains an element of L. Any such ball also contains an element outside of L since it contains an element outside of $K \cup L$ given that $p \in \partial(K \cup L)$, so we conclude that $p \in \partial L$ as desired.

Thus $\partial(K \cup L) \subseteq \partial K \cup \partial L$. Since K and L are compact, they are each closed in $\mathbb{R}^2$ so each contains their own boundary. Thus

$$\partial(K \cup L) \subseteq \partial K \cup \partial L \subseteq K \cup L,$$

showing that $K \cup L$ is closed since it contains its own boundary. Hence $K \cup L$ is closed and bounded, so it is compact.

(The portion in the middle which shows that $\partial(K \cup L) \subseteq \partial K \cup \partial L$ is tricky, and illustrates well how to work with various definitions.) $\square$

5. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 + x + \frac{x^2 y}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at $(0,0)$.

Bonus (3 extra points): Use the formal ϵ-δ definition of a limit to show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1,$$

thereby verifying that f is continuous at $(0,0)$.

Proof. We have

$$f(0,y) = 1 \quad \text{and} \quad f(x,0) = 1 + x.$$

The derivatives of these single-variable functions are the partial derivatives of f at $(0,0)$, so

$$\frac{\partial f}{\partial x}(0,0) = 1 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

Thus the Jacobian matrix of f at $(0,0)$ is $Df(0,0) = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

Setting $\mathbf{h} = (h,k)$, we have:

$$\frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - Df(\mathbf{0})\mathbf{h}}{\|\mathbf{h}\|} = \frac{1 + h + \frac{h^2 k}{\sqrt{h^2+k^2}} - 1 - \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix} \right) \left(\begin{smallmatrix} h \\ k \end{smallmatrix} \right)}{\sqrt{h^2+k^2}} = \frac{h^2 k}{h^2+k^2}.$$

In polar coordinates this becomes

$$r(\cos^2\theta + \sin\theta),$$

which converges to 0 as $(h,k) \to (0,0)$ by the squeeze theorem since

$$|r(\cos^2\theta + \sin\theta)| \leq 2r.$$

Thus

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - Df(\mathbf{0})\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0},$$

so f is differentiable at $(0,0)$ as claimed. $\square$

Proof of Bonus. Let $\epsilon > 0$ and let $\delta = \frac{\epsilon}{2}$. Then $\delta > 0$, and if need be we can shrink δ to guarantee that it is also less than 1. Suppose (x, y) satisfies

$$0 < \|(x, y)\| < \delta.$$

Then we also have

$$|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} < \delta$$

and

$$|y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} < \delta.$$

Thus:

$$|f(x, y) - 1| = \left| x + \frac{x^2 y}{\sqrt{x^2 + y^2}} \right| \leq |x| + \frac{|x|^2 |y|}{\sqrt{x^2 + y2}} \leq |x| + \frac{\sqrt{x^2 + y^2}^3}{\sqrt{x^2 + y^2}} = |x| + (x^2 + y^2) < \delta + \delta^2.$$

Since $\delta < 1$, $\delta^2 < \delta$ so we get

$$|f(x, y) - 1| < \delta + \delta^2 < 2\delta \leq \epsilon$$

by the choice of δ. Thus $0 < \|(x, y)\| < \delta$ implies $|f(x, y) - 1| < \epsilon$, which shows that

$$\lim_{(x,y) \to (0,0)} f(x, y) = 1$$

as required, and hence f is continuous at $(0, 0)$, which we already knew as a consequence of f being differentiable at $(0, 0)$. $\square$