Math 291-3: Midterm 2
Northwestern University, Spring 2016

Name: ______________________________

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) Any C^1 closed 1-form on the region obtained by removing $(1,1)$ from $\mathbb{R}^2$ is exact.

(b) If $f : \mathbb{R}^3 \to \mathbb{R}$ is C^2, then the 2-form

$$\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\right) dz \wedge dx + \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

is closed on $\mathbb{R}^3$.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose S is a smooth C^1 surface with parametrization

$$\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v)),\ \ (u,v) \in E$$

where E is a subset of $\mathbb{R}^2$, and let $\mathbf{c}(t) = (u(t), v(t)),\ \ a \le t \le b$ be a parametrization of a smooth C^1 curve in E. The composition $\mathbf{X} \circ \mathbf{c} : [a,b] \to \mathbb{R}^3$ then describes a smooth C^1 curve on S. Show that for any $t \in [a,b]$,

$$(\mathbf{X} \circ \mathbf{c})'(t) \cdot (\mathbf{X}_u \times \mathbf{X}_v)(u(t), v(t)) = 0.$$

Hint: Show that $(\mathbf{X} \circ \mathbf{c})'(t)$ is a linear combination of $\mathbf{X}_u(u(t), v(t))$ and $\mathbf{X}_v(u(t), v(t))$. (The point is that $(\mathbf{X} \circ \mathbf{c})'(t)$ gives a vector tangent to S at the point $\mathbf{X}(u(t), v(t))$, so this verifies that $(\mathbf{X}_u \times \mathbf{X}_v)(u(t), v(t))$ is orthogonal to every vector which is tangent to S at $\mathbf{X}(u(t), v(t))$, which is why $\mathbf{X}_u \times \mathbf{X}_v$ indeed gives vectors normal to S.)

3. (10 points) Suppose $\mathbf{F}, \mathbf{G}$ are two C^1 vector fields on $\mathbb{R}^3$ which are orthogonal at every point, meaning

$$\mathbf{F}(\mathbf{p}) \cdot \mathbf{G}(\mathbf{p}) = 0 \text{ for all } \mathbf{p} \in \mathbb{R}^3.$$

If C is a curve lying on a flow line of $\mathbf{G}$, determine the value of $\int_C \mathbf{F} \cdot d\mathbf{s}$.

4. (10 points) Compute

$$\int_C (2xy + yz)\,dx + (x^2 + xz - z - 2yz^3)\,dy + (y + xy - 3y^2z^2)\,dz$$

where C is the intersection of the plane $y = z$ with the cylinder $x^2 + z^2 = 1$, oriented counterclockwise when viewed from the positive z-axis.

5. (10 points) Suppose C is a smooth, C^1 curve in $\mathbb{R}^2$ which starts at $(4, 0)$, ends at $(-3, 0)$, and otherwise lies fully above the x-axis. Show that

$$\int_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = \pi.$$