Math 291-2: Midterm 2
Northwestern University, Winter 2016

Name: ____

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If a 2×2 matrix A only has 1 as an eigenvalue and $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\2\end{smallmatrix}\right)$ are both eigenvectors corresponding to 1, then A is diagonalizable.

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation, then T is differentiable. (Recall that T being affine means T is of the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some 2×2 matrix A and $\mathbf{b} \in \mathbb{R}^2$.)

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose A is a symmetric $n \times n$ matrix. Show that there exists a symmetric $n \times n$ matrix B such that $B^5 = A$. Hint: Diagonalization.

3. (10 points) For $k \neq 0$, determine the point(s) on the surface

$$-x^2 + y^2 - z^2 + 4xz = k$$

which are closest to $(0, 0, 0)$. (The answer will depend on k.) You should justify that your answers are correct, but doing so based on the shape of the surface is good enough.

4. (10 points) Suppose that K and L are compact subsets of $\mathbb{R}^2$. Show that their union $K \cup L$ is compact as well. Hint: To show $K \cup L$ is closed, first show that $\partial(K \cup L) \subseteq \partial K \cup \partial L$.

For grading purposes, 5 points will be given for showing that $K \cup L$ is closed and 5 points for showing it is bounded, so if you get stuck on one portion you should still do the other.

5. (10 points) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 + x + \frac{x^2 y}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at $(0,0)$.

Bonus (3 extra points): Use the formal ϵ-δ definition of a limit to show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1,$$

thereby verifying that f is continuous at $(0,0)$.