Math 300: Final Exam Solutions
Northwestern University, Spring 2017

1. Give an example of each of the following with brief justification.
(a) A true implication $P \Rightarrow Q$ whose converse $Q \Rightarrow P$ is false.
(b) A function $f : \mathbb{R} \to \mathbb{R}$ which is injective but not surjective.
(c) A countable subset of the power set of $\mathbb{R}$.

Solution. (a) The implication "If $x > 3$, then $x > 1$" is true, but the the converse "If $x > 1$, then $x > 3$" is not since $x = 2$ is a counterexample.

(b) The function f defined by $f(x) = e^x$ is injective since $e^x = e^y$ implies $x = y$, but it is not surjective since there is no x satisfying $e^x = 0$.

(c) The set $\{\{n\} \mid n \in \mathbb{N}\}$, which is the set whose elements are singletons $\{n\}$ for $n \in \mathbb{N}$, is a subset of $\mathcal{P}(\mathbb{R})$ since each $\{n\}$ is in $\mathcal{P}(\mathbb{R})$, and is countable since there are only countably many choices for n. □

2. (a) Show that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. You may take for granted the fact that for any $x \in \mathbb{R}$, there exists $M \in \mathbb{N}$ such that $x < M$.
(b) Show that if $x \leq y + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $x \leq y$.

Proof. (a) Let $\epsilon > 0$. Then $\frac{1}{\epsilon} \in \mathbb{R}$, so there exists $M \in \mathbb{N}$ such that $\frac{1}{\epsilon} < M$. Rearranging this inequality gives $\frac{1}{M} < \epsilon$, where the direction of the inequality is maintained since $M > 0$. This gives the desired result.

(b) By way of contrapositive, suppose $x > y$. Then $x - y > 0$, so by part (a) there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < x - y$. Then $x > y + \frac{1}{N}$, which justifies the contrapositive. □

3. Let A and B be sets. Show that

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

(This is known as the *symmetric difference* of A and B, and consists of all elements which belong to either A or B, but not both.)

Proof. Let $x \in (A \cup B) - (A \cap B)$. Then $x \in A \cup B$ and $x \notin A \cap B$. Since $x \in A \cup B$, we have that $x \in A$ or $x \in B$. Suppose $x \in A$. Since $x \notin A \cap B$, $x \notin A$ or $x \notin B$. But since we are assuming $x \in A$, it must be that $x \notin B$. Hence $x \in A - B$, so $x \in (A - B) \cup (B - A)$. Similarly, if $x \in B$, then $x \notin A \cap B$ implies that $x \notin A$, so $x \in B - A$ and again $x \in (A - B) \cup (B - A)$. Thus

$$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A).$$

Now let $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$; without loss of generality we may assume $x \in A - B$. Then $x \in A$ and $x \notin B$. Since $x \in A$, $x \in A \cup B$, and since $x \notin B$, $x \notin A \cap B$. Thus $x \in (A \cup B) - (A \cap B)$, so

$$(A \cup B) - (A \cap B) \supseteq (A - B) \cup (B - A).$$

Since both containments hold we conclude that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ as claimed. □

4. Suppose $f : A \to B$ is a function and that S is a subset of B.
(a) Show that $f(f^{-1}(S)) \subseteq S$.
(b) Show that if f is surjective, then $f(f^{-1}(S)) = S$.

Proof. (a) Let $y \in f(f^{-1}(S))$. Then there exists $x \in f^{-1}(S)$ such that $f(x) = y$. But by definition of $f^{-1}(S)$, $f(x) \in S$, so $y = f(x) \in S$. Hence $f(f^{-1}(S)) \subseteq S$.

(b) We need only show the backwards containment. Let $b \in S$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Since $f(a) = b \in S$, this means that $a \in f^{-1}(S)$, so $f(a) = b$ is actually in $f(f^{-1}(S))$. Thus $S \subseteq f(f^{-1}(S))$, and combined with part (a) we thus have equality. $\square$

5. Determine whether or not the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x, y, z) = (x + y + z, y + z, z)$$

is invertible.

Solution. This function is invertible. Indeed, we claim that $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$g(a, b, c) = (a - b, b - c, c)$$

is the inverse of f. We check:

$$g(f(x, y, z)) = g(x + y + z, y + z, z) = ([x + y + z] - [y + z], [y + z] - z, z) = (x, y, z),$$

so $g \circ f = id$, and

$$f(g(a, b, c)) = f(a - b, b - c, c) = ([a - b] + [b - c] + c, [b - c] + c, c) = (a, b, c),$$

so $f \circ g = id$ as well. Thus f is invertible with inverse g. $\square$

6. Define an equivalence relation on $\mathbb{R}$ by saying $x \sim y$ if $x - y \in \mathbb{Q}$. Determine, with justification, whether each equivalence class is countable or uncountable, and whether the set of equivalence classes is countable or uncountable.

Proof. Fix $x \in \mathbb{R}$. Then the equivalence class of x consists of all $y \in \mathbb{R}$ such that $x - y$ is some rational number. But this means that y is of the form $x +$ (rational), so

$$[x] = \{x + r \mid r \in \mathbb{R}\}.$$

Since $\mathbb{Q}$ is countable, there are only countably many choices for r, so there are only countably many elements in $[x]$. Hence each equivalence class $[x]$ is countable.

Now, the union of all equivalence classes is all of $\mathbb{R}$, so if there were only countably many equivalence classes

$$[x_1], [x_2], [x_3], \ldots,$$

we would have that

$$\mathbb{R} = [x_1] \cup [x_2] \cup [x_3] \cup \cdots$$

is a countable union of countable sets, so it would be countable itself. But $\mathbb{R}$ is uncountable, so there must be uncountably many equivalence classes, so the set of equivalence classes is uncountable. $\square$

7. A sequence $(r_1, r_2, r_3, \ldots)$ of rational numbers is *eventually constant* if there exists $r \in \mathbb{Q}$ and $N \in \mathbb{N}$ such that $r_n = r$ for all $n > N$. (In other words, all terms beyond some point are the same.) Show that the set of sequences of rational numbers which are eventually constant is countable.

Proof. For each $N \in \mathbb{N}$ and $r \in \mathbb{Q}$, let

$$S_{N,r} := \{(r_1, r_2, \ldots) \in \mathbb{Q}^\infty \mid r_n = r \text{ for } n > N\}.$$

So, $S_{N,r}$ is the set of sequences of rational numbers which are r beyond the N-th term. Such a sequence is thus fully characterized by the first N terms $r_1, \ldots, r_N$ and the number r, so the function $S_{N,r} \to \mathbb{Q}^{N+1}$ defined by

$$(r_1, r_2, \ldots, r_N, r, r, r, \ldots) \mapsto (r_1, r_2, \ldots, r_N, r)$$

is bijective. Since $\mathbb{Q}^{N+1}$ is a product of finitely many countable sets, it is countable so $S_{N,r}$ is countable as well. Hence, for each $r \in \mathbb{Q}$, the set

$$S_r := S_{1,r} \cup S_{2,r} \cup S_{3,r} \cup \cdots$$

of sequences which are eventually r is a countable union of countable sets, so it is countable. The set of eventually constant sequences is then the union of the sets S_r for varying r:

$$\bigcup_{r \in \mathbb{Q}} S_r,$$

so it too is a countable union of countable sets and is thus countable as well. □

8. Show that the set $\mathbb{Q}^\infty$ of *all* sequences $(r_1, r_2, r_3, \ldots)$ of rational numbers is uncountable by showing directly that given any infinite list of elements of $\mathbb{Q}^\infty$, there always exists an element of $\mathbb{Q}^\infty$ not included in that list. (Or in other words, given any function $\mathbb{N} \to \mathbb{Q}^\infty$, there exists an element of $\mathbb{Q}^\infty$ not included in its image.)

Proof. Let $f : \mathbb{N} \to \mathbb{Q}^\infty$ be any function. List the elements in the image as:

$$\begin{aligned} f(1) &= (r_{11}, r_{12}, r_{13}, \ldots) \\ f(2) &= (r_{21}, r_{22}, r_{23}, \ldots) \\ f(3) &= (r_{31}, r_{32}, r_{33}, \ldots) \\ &\vdots \end{aligned}$$

where each r_{ij} is in $\mathbb{Q}$. Define the sequence $(y_1, y_2, \ldots)$ by picking y_i to be different from r_{ii}; say:

$$y_i := \begin{cases} 3 & \text{if } r_{ii} \neq 3 \\ 5 & \text{if } r_{ii} = 3. \end{cases}$$

Then $(y_1, y_2, y_3, \ldots) \in \mathbb{Q}^\infty$ is not equal to any $f(n)$ since it differs from $f(n)$ in the n-th term, so it is not in the range of f. Hence f is not surjective, so no bijection between $\mathbb{N}$ and $\mathbb{Q}^\infty$ can exist, so $\mathbb{Q}^\infty$ is uncountable. □