Math 300: Final Exam Solutions
Northwestern University, Spring 2018

1. Give an example of each of the following with brief justification.
(a) An true implication $P \Rightarrow Q$ for which $\sim P \Rightarrow \sim Q$ is false.
(b) A function $f : [0,1] \to (0,1)$ which is surjective but not injective.
(c) A countably infinite number of points in $\mathbb{R}^2$.

Solution. (a) "If $x > 1$, then $x > 0$" is true since $1 > 0$, but "if $x \leq 1$, then $x \leq 0$" is false since $x = \frac{1}{2}$ is a counterexample.

(b) Define f by

$$f(x) = \begin{cases} x & x \neq 0, 1 \\ \frac{1}{2} & x = 0, 1 \end{cases}$$

This is surjective since for any $x \in (0,1)$, $f(x) = x$, but it is not injective since $f(0) = f(1)$.

(c) Take $\mathbb{Z} \times \mathbb{Z}$, which is all points (m, n) for which m and n are integers. This works since the product of two countably infinite sets is countably infinite. □

2. Let A and B be the following sets:

$$A = \left\{ n \in \mathbb{Z} \;\middle|\; n = 8k^2 + 15 \text{ for some } k \in \mathbb{Z} \right\}$$

and

$$B = \{ n \in \mathbb{Z} \mid n = 4k + 3 \text{ for some } k \in \mathbb{Z} \}.$$

Show that $A \subseteq B$ and $A \neq B$.

Proof. Let $n \in A$. Then there exists $k \in \mathbb{Z}$ such that $n = 8k^2 + 15$. This can be written as:

$$n = 8k^2 + 15 = 4(2k^2 + 2) + 3,$$

so for the integer $\ell = 2k^2 + 2$ we do have $n = 4\ell + 3$. Thus $n \in B$, so $A \subseteq B$.

Now, $7 \in B$ since $7 = 4(1) + 3$. But if there were to exist $k \in \mathbb{Z}$ for which $7 = 8k^2 + 15$, we would have $k^2 = -1$, which is not possible. Hence 7 cannot be expressed in the form required of an element of A, so $7 \in A$ and thus $A \neq B$ as required. □

3. (a) Determine the following union and prove that your answer is correct.

$$\bigcup_{n \in \mathbb{N}} \left(\tfrac{1}{n}, n\right)$$

(b) Determine the following intersection and prove that your answer is correct.

$$\bigcap_{a < 0} (a, 1]$$

Solution. (a) We claim that this union equals $(0, \infty)$. Indeed, let $x \in \bigcup_{n \in \mathbb{N}} (\frac{1}{n}, n)$. Then $x \in (\frac{1}{n}, n)$ for all $n \in \mathbb{N}$, so

$$0 < \frac{1}{n} < x < n < \infty$$

and hence $x \in (0, \infty)$. Conversely let $x \in (0, \infty)$. By the Archimedean Property of $\mathbb{R}$ there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\frac{1}{n_1} < x \text{ and } x < n_2.$$

Then $N := \max\{n_1, n_2\}$ satisfies $\frac{1}{N} \leq \frac{1}{n_1} < x < n_2 \leq N$, so $x \in (\frac{1}{N}, N)$. Thus $x \in \bigcup_{n\in\mathbb{N}}(\frac{1}{n}, n)$, so we conclude that $\bigcup_{n\in\mathbb{N}}(\frac{1}{n}, n) = (0, \infty)$.

(b) We claim that this intersection equals $[0, 1]$. First, let $x \in [0, 1]$. Then for any $a < 0$, we have

$$a < 0 \leq x \leq 1,$$

so $x \in (a, 1]$. Thus $x \in \bigcap_{a<0}(a, 1]$.

Now, let $x \in \bigcap_{a<0}(a, 1]$, which means that

$$a < x \leq 1 \text{ for all } a < 0.$$

Certainly we have $x \leq 1$. To show that $0 \leq x$, we prove the following contrapositive: if $x < 0$, then there exists $a < 0$ such that $x \leq a$. Indeed, if $x < 0$, then $a = \frac{x}{2}$ is negative and satisfies $x \leq a$, so the contrapositive is proved. Thus $a < x$ for all $a < 0$ implies $0 \leq x$, so we conclude that the given $x \in \bigcap_{a<0}(a, 1]$ is in $[0, 1]$ as well, and hence $\bigcap_{a<0}(a, 1] = [0, 1]$ as claimed. □

4. Suppose $f : A \to B$ is a function and that S is a subset of A.

(a) Show that $S \subseteq f^{-1}(f(S))$.

(b) Show that if f is injective, then $S = f^{-1}(f(S))$.

Proof. (a) Let $x \in S$. Then $f(x) \in f(S)$ by definition of image. But then x is sent to something in $f(S)$, so $x \in f^{-1}(f(S))$ by definition of preimage. Hence $S \subseteq f^{-1}(f(S))$.

(b) Let $x \in f^{-1}(f(S))$. Then $f(x) \in f(S)$ so there exists $s \in S$ such that $f(x) = f(s)$. But since f is now injective, we have $x = s$, so $x \in S$. Hence $f^{-1}(f(S)) \subseteq S$ in this case, and thus we have equality as claimed. □

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x, y) = (2x + y, x + 2y).$$

Show that f is invertible by finding its inverse and verifying that it is indeed the inverse.

Solution. We claim that the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$g(x, y) = (\tfrac{2}{3}x - \tfrac{1}{3}y, -\tfrac{1}{3}x + \tfrac{2}{3}y)$$

is the inverse of f, which we now verify. First, we compute:

$$\begin{aligned} f(g(x, y)) &= f(\tfrac{2}{3}x - \tfrac{1}{3}y, -\tfrac{1}{3}x + \tfrac{2}{3}y) \\ &= (2[\tfrac{2}{3}x - \tfrac{1}{3}y] + [-\tfrac{1}{3}x + \tfrac{2}{3}y], [\tfrac{2}{3}x - \tfrac{1}{3}y] + 2[-\tfrac{1}{3}x + \tfrac{2}{3}y]) \\ &= (\tfrac{3}{3}x, \tfrac{3}{3}y) \\ &= (x, y) \end{aligned}$$

so $f \circ g$ is the identity function on $\mathbb{R}^2$. Next:

$$g(f(x, y)) = g(2x + y, x + 2y)$$

$$
\begin{aligned}
&= (\tfrac{2}{3}[2x+y] - \tfrac{1}{3}[x+2y], -\tfrac{1}{3}[2x+y] + \tfrac{2}{3}[x+2y]) \\
&= (\tfrac{3}{3}x, \tfrac{3}{3}y) \\
&= (x,y)
\end{aligned}
$$

so $g \circ f$ is also the identity, and thus f is invertible with inverse g as claimed. $\square$

6. Define an equivalence relation on $\mathbb{R}^2$ by saying

$$(x,y) \sim (a,b) \text{ if there exists } \lambda \neq 0 \text{ such that } a = \lambda x \text{ and } b = \lambda y.$$

Show that the set of equivalence classes has the same cardinality as $\mathbb{R}$. (Careful: this is not asking about the cardinality of each equivalence class, but rather of the set whose **elements** are the equivalence classes.)

Proof. Visually, for non-origin points (x,y) and (a,b), $(x,y) \sim (a,b)$ if (a,b) and (x,y) lie on the same line through the origin in $\mathbb{R}^2$, while the origin $(0,0)$ is equivalent to only itself. Thus, the equivalence class of any non-origin point intersects the upper half of the unit circle in some point whose polar angle θ lies in the interval $[0,\pi)$; if we want to be precise, if (x,y) is not the origin, we are taking the angle $\theta = \tan^{-1}(\frac{y}{x})$, which we interpret to be $\frac{\pi}{2}$ when $x = 0$.

Any non-origin equivalence class is uniquely characterized by this polar angle, so the function $f : \{\text{equivalence classes of } \sim) \to [0,\pi]$ defined by

$$f : [(x,y)] \to \begin{cases} \text{angle in } [0,\pi) \text{ the line } [(x,y)] \text{ makes with the upper semi-circle} & (x,y) \neq (0,0) \\ \pi & (x,y) = (0,0) \end{cases}$$

is a bijection. The value π was included in order to have something the equivalence class of $(0,0)$ could be sent to, and f is injective since if non-origin points (x,y) and (a,b) determine the same angle as described above, then they must lie on the same line through the origin, which makes them equivalent and hence $[(x,y)] = [(a,b)]$. The function f is surjective since given an angle $\theta \in [0,\pi)$, the points $(\cos\theta, \sin\theta)$ is one whose equivalence class is sent to θ.

Thus, the set of equivalence classes of $\sim$ has the same cardinality as the interval $[0,\pi]$, which in turn has the same cardinality as $\mathbb{R}$, so the set of equivalence classes has the same cardinality as $\mathbb{R}$ as well. (The point of this problem, as usual, was to determine how to uniquely characterize an equivalence classes via some real number, in this case via an angle. The grading was fairly lenient in terms of getting all the details right! Another approach could have been to try to characterize and equivalence class by the *slope* of the line it represents.) $\square$

7. Let F be the set of all finite subsets of $\mathbb{Q}$:

$$F = \{S \subseteq \mathbb{Q} \mid S \text{ is finite}\}.$$

Show that F is countable. Hint: For a fixed $n \geq 0$, how many subsets of $\mathbb{Q}$ have at most n elements?

Proof. For a fixed $n \geq 0$, let F_n denote the set of subsets of $\mathbb{Q}$ which have n elements. Then F_0 has only the empty set in it, and for $n > 1$ a set in F_n looks like $\{a_1, \ldots, a_n\}$ with $a_i \in \mathbb{Q}$—let us assume that these elements are always listed in increasing order: $a_1 < a_2 < \ldots < a_n$. For $n > 1$, the function

$$F_n \to \mathbb{Q}^n \text{ defined by } \{a_1, \ldots, a_n\} \mapsto (a_1, \ldots, a_n)$$

is then injective, so since $\mathbb{Q}^n$ is countable we have that F_n is countable as well. Note that F_0 is also countable.

Our given set F is the union of all the F_n:

$$F = \bigcup_{n \geq 0} F_n,$$

so since this is the union of countably many countable sets, it is countable itself as claimed. $\square$

8. Suppose S is a finite set with at least two elements. Show that

$$\underbrace{S \times S \times S \times \cdots}_{\text{countably infinitely many}}$$

is uncountable. To be clear, elements in this set look like infinite sequences

$$(x_1, x_2, x_3, \ldots)$$

where each x_i is in S. Also, what is the cardinality of this set when $|S| \leq 1$?

Solution. Let $a, b \in S$ be two distinct elements. Denote the set in question by S^∞, and let $f : \mathbb{N} \to S^\infty$ be any function. Write the elements in the image of f as:

$$\begin{aligned} f(1) &= (x_{11}, x_{12}, \ldots) \\ f(2) &= (x_{21}, x_{22}, \ldots) \\ \vdots\ &\ \ \vdots \end{aligned}$$

where each x_{ij} is in S. Define the element $y = (y_1, y_2, \ldots)$ of S^∞ by setting

$$y_i = \begin{cases} a & \text{if } x_{ii} = b \\ b & \text{if } x_{ii} \neq b \end{cases}$$

Then for any $n \in \mathbb{N}$, $y \neq f(n)$ since the sequences defining y and $f(n)$ differ in the n-term since if one is b the other is a, and a and b are distinct. Hence y is no in the image of f, so f cannot be surjective. Thus no bijection $\mathbb{N} \to S^\infty$ can exist, so since S is infinite we conclude that S is uncountable.

If S is empty, S^∞ is also empty and has cardinality 0, while if S has only one element, so does S^∞ and thus has cardinality 1. $\square$