Math 300: Final Exam Solutions
Northwestern University, Winter 2019

1. Give an example of each of the following with brief justification.
(a) A function $f : \mathbb{R} \to \mathbb{R}$ which is surjective but not injective.
(b) Nonempty subsets A_n of $\mathbb{Z}$ indexed by $n \in \mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}} A_n$ is empty.
(c) A subset of $\mathbb{R}$ not containing π but whose supremum is π.

Solution. (a) The function

$$f(x) = \begin{cases} \ln x & x > 0 \\ 0 & x \leq 0 \end{cases}$$

works. This is surjective since given any $y \in \mathbb{R}$, $e^y \in \mathbb{R}$ satisfies $f(e^y) = y$, and it is not injective since many things get sent to 0.

(b) For each $n \in \mathbb{N}$ set $A_n = \{n\}$. Each A_n contains only one element, and this element is different for different n, so the intersection of all these sets is indeed empty.

(c) The interval $(0, \pi)$ works. This certainly does not contain π, but the fact that its elements get arbitrarily close to π imply that the supremum is π. (To be more precise, π is an upper bound of this interval and given any $\epsilon > 0$, we can find an element in this interval which is larger than $\pi - \epsilon$, so nothing smaller than π can be an upper bound.) □

2. Let A and B be the following sets:

$$A = \{n \in \mathbb{Z} \mid n = 7k - 17 \text{ for some } k \in \mathbb{Z}\}$$

and

$$B = \left\{n \in \mathbb{Z} \;\middle|\; n = 14k^3 + 4 \text{ for some } k \in \mathbb{Z}\right\}.$$

Show that $B \subseteq A$ and $A \neq B$.

Proof. Let $n \in B$. Then there exists $k \in \mathbb{Z}$ such that $n = 14k^3 + 4$. We can rewrite this expression as

$$n = 14k^3 + 4 = 7(2k^2 + 3) - 17,$$

which since $2k^2 + 3 \in \mathbb{Z}$ shows that n is of the form required of an element in A. Hence $n \in A$ so $B \subseteq A$.

Note that $-17 \in A$ since $-17 = 7(0) - 17$. If $-17 \in B$, then there exists $k \in \mathbb{Z}$ such that $-17 = 14k^3 + 4$, which requires that $-21 = 14k^3$ and hence $-\frac{3}{2} = k^3$. But there is no integer which satisfies this condition, so -17 cannot be in B. Hence -17 is an element of A which is not in B, so $A \neq B$. □

3. (a) Prove the following set containment:

$$[0, 1) \subseteq \bigcup_{\epsilon>0} [0, 1 - \epsilon)$$

(b) Prove the following set containment:

$$\bigcap_{\epsilon>0} [0, 1 + \epsilon] \subseteq [0, 1]$$

Proof. (a) Let $x \in [0,1)$. Set $\epsilon' = \frac{1-x}{2}$, which is positive since $x < 1$. Then

$$\epsilon' = \frac{1-x}{2} < 1 - x, \text{ so } 0 \leq x < 1 - \epsilon'.$$

Hence $x \in [0, 1-\epsilon')$ for this specific $\epsilon = \epsilon'$, so $x \in \bigcup_{\epsilon>0} [0, 1-\epsilon)$ and thus $[0,1)$ is a subset of this union as claimed. (The value of ϵ' we used was found by working backwards from the inequality $x < 1 - \epsilon$ we wanted to obtain.)

(b) To say that $x \in \bigcap_{\epsilon>0} [0, 1+\epsilon]$ is to say that $x \in [0, 1+\epsilon]$ for all $\epsilon > 0$, or in other words that $0 \leq x \leq 1 + \epsilon$ for all $\epsilon > 0$. Thus we must show: if $x \in [0, 1+\epsilon]$ for all $\epsilon > 0$, then $x \in [0,1]$.

For this we prove instead the contrapositive: if $x \notin [0,1]$, then there exists $\epsilon > 0$ such that $x \notin [0, 1+\epsilon]$. So, suppose $x \notin [0,1]$. Then either $x < 0$ or $1 < x$. In the first case then for sure $x \in [0, 1+\epsilon]$ regardless of the value of ϵ, so there is nothing left to do in this case. So suppose $1 < x$ and set $\epsilon = \frac{x-1}{2}$, which is positive since $1 < x$. Then

$$\epsilon = \frac{x-1}{2} < x - 1, \text{ so } 1 + \epsilon < x$$

and hence $x \notin [0, 1+\epsilon]$. This proves the required contrapositive, so we conclude that the given intersection is a subset of $[0,1]$ as claimed. (The ϵ value we used in the contrapositive was found by working backwards from the inequality $1 + \epsilon < x$ we needed in order to guarantee that $x \notin [0, 1+\epsilon]$.) □

4. Suppose $f : A \to B$ is a function and that $X, Y \subseteq A$.

(a) Show that $f(X) - f(Y) \subseteq f(X - Y)$.

(b) Show that if f is injective, then $f(X - Y) \subseteq f(X) - f(Y)$.

Proof. (a) Let $b \in f(X) - f(Y)$. Then $b \in f(X)$ and $b \notin f(Y)$. Since $b \in f(X)$, there exists $x \in X$ such that $f(x) = b$. But since $b \notin f(Y)$, there is nothing in Y which is sent to b, so in particular this x cannot be in Y. Hence $x \in X - Y$, so there is an element of $X - Y$ which is sent to b, meaning that $b \in f(X - Y)$. Hence $f(X) - f(Y) \subseteq f(X - Y)$.

(b) Let $b \in f(X - Y)$. Then there exists $x \in X - Y$ such that $f(x) = b$. Since $x \in X - Y$, $x \in X$ and $x \notin Y$. Since $x \in X$, we have $b = f(x) \in f(X)$. But since f is injective, there are no elements apart from x which are sent to b, so since this specific x is not in Y, there is hence nothing in Y which is sent to b. Thus $b \notin f(Y)$, so $b \in f(X) - f(Y)$, and we conclude that $f(X - Y) \subseteq f(X) - f(Y)$. (The injectivity of f was needed to guarantee that since $f(x) = b$ and $x \notin Y$, then $b \notin f(Y)$. If f were not injective the issue is that even though $x \notin Y$, there could still be some element of Y not equal to x which was also sent to b, in which case b would be in $f(Y)$.) □

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x, y) = (2x - y - 1, x + 3).$$

(a) Show that f is surjective.

(b) Let S denote the subset of $\mathbb{R}^2$ consisting of all points on the line $y = x$. Show that the preimage $f^{-1}(S)$ of S under f is also a line by finding an explicit equation of this line. Just write the equation down after you work it out—you do not have to prove formally that $f^{-1}(S)$ equals the line you find by showing that each is a subset of the other.

Proof. (a) Let $(a,b) \in \mathbb{R}^2$. Then

$$f(b-3, 2b-a-7) = (2[b-3] - [2b-a-7] - 1, [b-3]+3) = (a,b),$$

which shows that f is surjective. (The input point $(b-3, 2b-a-7)$ was found by determining what x and y should be in order to have $f(x,y) = (2x-y-1, x+3) = (a,b)$.)

(b) The preimage of the line S consists of all inputs (x,y) for which $f(x,y) = (2x-y-1, x+3)$ is one this line. But this requires that the two coordinates of $(2x-y-1, x+3)$ be the same, which gives the condtion

$$2x - y - 1 = x + 3.$$

This is the same as $y = x - 4$, so the preimage of S is the line with equation $y = x - 4$. ☐

6. Define an equivalence relation on $\mathbb{R}^2$ by saying

$$(x,y) \sim (a,b) \text{ if } 2(y-b) = -3(x-a).$$

Show that the set of equivalence classes has the same cardinality as $\mathbb{R}$. (Careful: this is not asking about the cardinality of each equivalence class, but rather of the set whose **elements** are the equivalence classes.) Hint: Think about what each equivalence class looks like geometrically, and how you can characterize an entire class using a single real number.

Proof. The equivalence class of $[(x,y)]$ consists of all points (a,b) satisfying

$$2(y-b) = -3(x-a).$$

But this is the same as saying that (a,b) lies on the line of slope $-\frac{3}{2}$ which passes through (x,y). So, each equivalence class is a line of slope $-\frac{3}{2}$, and we can uniquely characterize such a line using its x-intercept. For the line defined by $[(x,y)]$, this x-intercept (i.e. the point a such that $(a,0)$ on this line) is $\frac{1}{3}(2y+3x)$, so this number uniquely characterizes the entire equivalence class $[(x,y)]$.

If your answer said something along the lines of "there are as many equivalence classes as there are possible choices for x-intercepts, and there as many choices for this as there are real numbers", chances are you got full credit. But let us now be completely precise: we claim that the function from the set of equivalence classes to $\mathbb{R}$ defined by

$$f : [(x,y)] \mapsto \frac{1}{3}(2y+3x)$$

is a bijection. First, this function is well-defined in the sense that if (a,b) gives the same equivalence class as (x,y), then the value $\frac{1}{3}(2b+3a)$ we get from this second point is the same as the value $\frac{1}{3}(2y+3x)$ from the equivalent point (x,y), so that the value of this function really only does depend on the equivalence class $[(x,y)]$ and not the specific point we choose to represent this equivalence class. Indeed, if $[(a,b)] = [(x,y)]$, then $(x,y) \sim (a,b)$, so $2(y-b) = -3(x-a)$ and rearranging terms and dividing by 3 does give

$$\frac{1}{3}(2y+3x) = \frac{1}{3}(2b+3a)$$

so that plugging in $[(x,y)]$ and $[(a,b)]$ into f give the same output.

Now, f is surjective since given $x \in \mathbb{R}$, $(x,0) \mapsto \frac{1}{3}(2(0) = 3x) = x$, or in other words the equivalence class of $(x,0)$ is sent to x. To see that f is injective, suppose $[(x,y)]$ and $[(a,b)]$ get sent to the same value, meaning that

$$\frac{1}{3}(2y+3x) = \frac{1}{3}(2b+3a).$$

Then $2y + 3x = 2b + 3a$, so $2(y - b) = -3(x - a)$. Hence $(x, y) \sim (a, b)$, so $[(x, y)] = [(a, b)]$ as desired. Thus f is bijective, so the set of equivalence classes has the same cardinality as $\mathbb{R}$. $\square$

7. A function $f : \mathbb{N} \to \mathbb{N}$ is said to be *periodic* if there exists $N \in \mathbb{N}$ such that

$$f(x + N) = f(x) \text{ for all } x \in \mathbb{N}.$$

(So, the values of f begin to repeat after some point: $f(N+1) = f(1), f(N+2) = f(2)$, etc.) Show that the set of periodic functions from $\mathbb{N}$ to $\mathbb{N}$ is countable.

Hint: For a fixed $N \in \mathbb{N}$, consider the set X_N of functions from $\mathbb{N}$ to $\mathbb{N}$ satisfying the periodic condition $f(x+N) = f(x)$ for that specific N. Show that each X_N is countable, making use of the fact that such a function is characterized by the values of $f(1), f(2), \ldots, f(N)$ since other values beyond this are determined by the periodic condition.

Proof. Fix $N \in \mathbb{N}$ and let X_N denote the set of functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(x + N) = f(x)$ for all $x \in \mathbb{N}$. For such a function f, the values of $f(1), f(2), \ldots, f(N)$ can be anything in $\mathbb{N}$, but once these are specified all other values are determined since $f(N + 1) = f(1), f(N + 2) = f(2), \ldots, f(2N + 1) = f(1), f(2N + 2) = f(2), \ldots$ and so on. This implies that the function $X_N \to \mathbb{N}^N$ defined by

$$f \mapsto (f(1), f(2), \ldots, f(N))$$

is bijective. Indeed, it is injective since if $f, g \in X_N$ satisfy

$$f(1) = g(1), f(2) = g(2), \ldots, f(N) = g(N),$$

then we must have $f(x) = g(x)$ for all $x \in \mathbb{N}$ by the periodic condition, so that f and g are the same function. The function above is surjective since given any $(n_1, \ldots, n_N) \in \mathbb{N}^N$, we can define a periodic function $f : \mathbb{N} \to \mathbb{N}$ which has these as its initial values by defining

$$f(1) = n_1,\ f(2) = n_2,\ f(3) = n_3, \ldots f(N) = n_N$$

and then using the fact that $f(x + N)$ should be equal to $f(x)$ for all x to define the values of f for inputs beyond N.

Thus we find that X_N has the same cardinality as $\mathbb{N}^N$, and since the latter is a product of finitely many countable sets, it is countable and so X_N is countable for each $N \in \mathbb{N}$. The set of all periodic functions $\mathbb{N} \to \mathbb{N}$ is the union of these X_N:

$$\{\text{periodic functions } \mathbb{N} \to \mathbb{N}\} = \bigcup_{n \in \mathbb{N}} X_n,$$

so since this is a union of countably many countable sets, it is countable as well. $\square$

8. Show that the set of *all* functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable. Hint: A function $f : \mathbb{N} \to \mathbb{N}$ is determined by the infinite sequence containing its values:

$$(f(1), f(2), f(3), \ldots)$$

which is an element of $\mathbb{N}^\infty$. Thus, the set of functions from $\mathbb{N}$ to $\mathbb{N}$ has the same cardinality as $\mathbb{N}^\infty$, which can you take for granted. So, in other words, the problem is really to show that $\mathbb{N}^\infty$ is uncountable, which you must do directly without quoting any result from class.

Proof. A function $f : \mathbb{N} \to \mathbb{N}$ is determined by the sequence of its values:

$$(f(1), f(2), f(3), \ldots).$$

Indeed, if f and g are two functions which give the same sequence of values, then $f(n) = g(n)$ for all $n \in \mathbb{N}$ so that f and g are the same function. And, given any sequence $(n_1, n_2, n_3, \ldots)$ of natural numbers, there is a function $f : \mathbb{N} \to \mathbb{N}$ which has precisely these values since we can define $f(k)$ to be n_k for all $k \in \mathbb{N}$. This shows that the set of functions $\mathbb{N} \to \mathbb{N}$ has the same cardinality as the set $\mathbb{N}^\infty$ of such sequences, or more precisely that the function from the set of functions $\mathbb{N} \to \mathbb{N}$ to $\mathbb{N}^\infty$ defined by

$$f \mapsto (f(1), f(2), f(3), \ldots)$$

is bijective. Thus, to show that the set of functions $\mathbb{N} \to \mathbb{N}$ is uncountable we need only show that $\mathbb{N}^\infty$ is uncountable.

Let $h : \mathbb{N} \to \mathbb{N}^\infty$ be any function and list the elements in its image in the following way:

$$\begin{aligned} h(1) &= (x_{11}, x_{12}, x_{13}, \ldots) \\ h(2) &= (x_{21}, x_{22}, x_{23}, \ldots) \\ h(3) &= (x_{31}, x_{32}, x_{33}, \ldots) \\ \vdots \; & \; \vdots \end{aligned}$$

Define $y = (y_1, y_2, y_3, \ldots) \in \mathbb{N}^\infty$ by setting

$$y_n = \begin{cases} 4 & \text{if } x_{nn} = 5 \\ 5 & \text{if } x_{nn} \neq 5 \end{cases}$$

Then $y \neq h(n)$ for any $n \in \mathbb{N}$ since the sequence defining y and the sequence given by $f(n)$ differ in their n-th terms. Hence y is not in the image of h, so h is not surjective. Thus no function from $\mathbb{N}$ to $\mathbb{N}^\infty$ can be surjective, so not such function can be bijective either. Since $\mathbb{N}^\infty$ is infinite but not countable, we conclude that it is uncountable, and hence the set of functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable too. □