Math 300: Final Exam
Northwestern University, Spring 2018

Name: ____

1. (10 points) Give an example of each of the following with brief justification.
(a) An true implication $P \Rightarrow Q$ for which $\sim P \Rightarrow \sim Q$ is false.
(b) A function $f : [0, 1] \to (0, 1)$ which is surjective but not injective.
(c) A countably infinite number of points in $\mathbb{R}^2$.

Problem	Score
1	
2	
3	
4	
5	
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7	
8	
Total	

2. (10 points) Let A and B be the following sets:

$$A = \left\{ n \in \mathbb{Z} \;\middle|\; n = 8k^2 + 15 \text{ for some } k \in \mathbb{Z} \right\}$$

and

$$B = \{ n \in \mathbb{Z} \mid n = 4k + 3 \text{ for some } k \in \mathbb{Z} \}.$$

Show that $A \subseteq B$ and $A \neq B$.

3. (a) (5 points) Determine the following union and prove that your answer is correct.

$$\bigcup_{n\in\mathbb{N}} \left(\tfrac{1}{n}, n\right)$$

(b) (5 points) Determine the following intersection and prove that your answer is correct.

$$\bigcap_{a<0} (a, 1]$$

4. Suppose $f : A \to B$ is a function and that S is a subset of A.

(a) (5 points) Show that $S \subseteq f^{-1}(f(S))$.

(b) (5 points) Show that if f is injective, then $S = f^{-1}(f(S))$.

5. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x, y) = (2x + y, x + 2y).$$

Show that f is invertible by finding its inverse and verifying that it is indeed the inverse.

6. (10 points) Define an equivalence relation on $\mathbb{R}^2$ by saying

$$(x, y) \sim (a, b) \text{ if there exists } \lambda \neq 0 \text{ such that } a = \lambda x \text{ and } b = \lambda y.$$

Show that the set of equivalence classes has the same cardinality as $\mathbb{R}$. (Careful: this is not asking about the cardinality of each equivalence class, but rather of the set whose **elements** are the equivalence classes.)

7. (10 points) Let F be the set of all finite subsets of $\mathbb{Q}$:

$$F = \{S \subseteq \mathbb{Q} \mid S \text{ is finite}\}.$$

Show that F is countable. Hint: For a fixed $n \geq 0$, how many subsets of $\mathbb{Q}$ have at most n elements?

8. (10 points) Suppose S is a finite set with at least two elements. Show that

$$\underbrace{S \times S \times S \times \cdots}_{\text{countably infinitely many}}$$

is uncountable. To be clear, elements in this set look like infinite sequences

$$(x_1, x_2, x_3, \dots)$$

where each x_i is in S. Also, what is the cardinality of this set when $|S| \leq 1$?