Math 300: Midterm 1 Solutions
Northwestern University, Spring 2018

1. Give an example of each of the following with brief justification.

(a) A false implication (i.e. "if P, then Q" statement) whose converse is true.

(b) Subsets A, B of $\mathbb{R}$ such that $\mathbb{R}\backslash(A \cup B)$ is the empty set.

Solution. (a) The implication "If $x > 4$, then $x > 2$" is true buts its converse "If $x > 2$, then $x > 4$" is false since $x = 3$ provides a counterexample.

(b) Take $A = (-\infty, 0]$ and $B = [0, \infty)$. Then $A \cup B = \mathbb{R}$, so $\mathbb{R}\backslash(A \cup B)$ is empty. $\square$

2. Let $n \in \mathbb{Z}$. Show that 9 divides n if and only if 9 divides $4n$. You may use basic properties of even and odd integers (i.e. what happens when you multiply two odd integers together, two even integers together, or an odd with an even), but no other properties of relatively prime integers. For instance, saying something along the lines of "if $4n$ is divisible by 9, then n is divisible by 9 since 4 and 9 are relatively prime" is not enough.

Proof. Suppose 9 divides n. Then there exists $k \in \mathbb{Z}$ such that $n = 9k$. This gives $4n = 9(4k)$, so 9 divides $4n$ as well.

Conversely suppose 9 divides $4n$. Then there exists $k \in \mathbb{Z}$ such that $4n = 9k$. This k must be even, since if it were odd $9k = 4n$ would be odd as well, which it is not. Hence $k = 2\ell$ for some $\ell \in \mathbb{Z}$. Then $4n = 9k$ gives $4n = 9(2\ell)$, or $2n = 9\ell$. Again this ℓ cannot be odd since then $9\ell = 2n$ would be odd, so there exists $b \in \mathbb{Z}$ such that $\ell = 2b$. Then $n = 9b$, so 9 divides n as required. $\square$

3. Let A, B, C be subsets of some larger set. Show that $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$.

Proof. Let $x \in A\backslash(B \cap C)$. Then $x \in A$ and $x \notin B \cap C$. Then $x \notin B \cap C$, we have $x \notin B$ or $x \notin C$. Without loss of generality, assume $x \notin B$. Then $x \in A\backslash B$, so $x \in (A\backslash B) \cup (A\backslash C)$. Hence $A\backslash(B \cap C) \subseteq (A\backslash B) \cup (A\backslash C)$.

Conversely suppose $x \in (A\backslash B) \cup (A\backslash C)$. Then $x \in A\backslash B$ or $x \in \backslash C$. Without loss of generality we may assume $x \in A\backslash B$. Then $x \in A$ and $x \notin B$. Since $x \notin B$, $x \notin B \cap C$. Thus $x \in A\backslash(B \cap C)$, so $(A\backslash B) \cup (A\backslash C) \subseteq A\backslash(B \cap C)$. Hence we equality as claimed. $\square$

4. At no point in either part below can you take some kind of limit. You must find a different way.

(a) Let $x \in \mathbb{R}$. Show that if $x < 5 + \frac{1}{n^2}$ for all $n \in \mathbb{N}$, then $x \leq 5$.

(b) Determine the following intersection and prove that your answer is correct.

$$\bigcap_{n \in \mathbb{N}} [-1, 5 + \tfrac{1}{n^2})$$

Solution. (a) We prove the contrapositive: if $x > 5$, then there exists $n \in \mathbb{N}$ such that $x \geq 5 + \frac{1}{n^2}$. Suppose $x > 5$. Then $x - 5 > 0$, so by the Archimedean Property of $\mathbb{R}$ there exists $N \in \mathbb{N}$ such that

$$\frac{1}{N} < x - 5.$$

Since $N \geq N^2$, $\frac{1}{N^2} \leq \frac{1}{N}$, so we have $\frac{1}{N^2} < x - 5$ as well. Hence

$$x > 5 + \frac{1}{N^2},$$

as required in the contrapositive.

(b) We claim that the given intersection equals $[-1, 5]$. First, if $x \in [-1, 5]$, then $-1 \leq x \leq 5$ so

$$-1 \leq x \leq 5 < 5 + \frac{1}{n^2} \text{ for all } n \in \mathbb{N}.$$

Hence $x \in [-1, 5 + \frac{1}{n^2})$ for all $n \in \mathbb{N}$, so $x \in \bigcap_{n \in \mathbb{N}} [-1, 5 + \frac{1}{n^2})$.

Now suppose $x \in \bigcap_{n \in \mathbb{N}} [-1, 5 + \frac{1}{n^2})$. Then $x \in [-1, 5 + \frac{1}{n^2})$ for all $n \in \mathbb{N}$, so in particular

$$x < 5 + \frac{1}{n^2} \text{ for all } n \in \mathbb{N}.$$

Part (a) then implies that $x \leq 5$, so $-1 \leq x \leq 5$. Thus $x \in [-1, 5]$, so $\bigcap_{n \in \mathbb{N}} [-1, 5 + \frac{1}{n^2}) = [-1, 5]$ as claimed. $\square$

5. Suppose $A \subseteq \mathbb{R}$ has a supremum, and let $4 + A$ denote the set of all numbers obtained by adding 4 to elements of A:

$$4 + A := \{4 + a \mid a \in A\}.$$

Show that $\sup(4 + A) = 4 + \sup A$. (Note: $4 + A$ is simply notation we use to the describe the set in question; we are not literally adding the number 4 to the set A.)

Proof. Let $x \in 4 + A$. Then $x = 4 + a$ for some $a \in A$. Since $\sup A$ is an upper bound of A, $a \leq \sup A$, so

$$x = 4 + a \leq 4 + \sup A.$$

Thus $4 + \sup A$ is an upper bound of $4 + A$.

Now, suppose u is any upper bound of $4 + A$. Then for any $a \in A$, $4 + a \in 4 + A$ so

$$4 + a \leq u.$$

Rearranging gives $a \leq u - 4$, and since $a \in A$ was arbitrary this shows that $u - 4$ is an upper bound of A. By definition of $\sup A$, we thus have $\sup A \leq u - 4$, so $4 + \sup A \leq u$. Thus $4 + \sup A$ is an upper bound of $4 + A$ which is smaller than or equal to any other upper bound of $4 + A$, so it is the supremum of $4 + A$ as claimed. $\square$