Math 300: Midterm 1 Solutions
Northwestern University, Winter 2019

1. Give an example of each of the following with brief justification.

(a) (5 points) An upper bound of $A := \{x \in \mathbb{R} \mid x^3 \leq 3\}$ which is not the supremum of A.

(b) (5 points) A subset A of $\mathbb{R}$ and a subset B of $\mathbb{R}^2$ such that $\mathbb{R}^2 - (A \times A) \neq B$.

Solution. (a) The number $10^3 = 1000$ is an upper bound of A, since

$$x^3 \leq 3 \leq 10^3 \text{ implies } x \leq 10.$$

But 9 is also an upper bound since $x^3 \leq 3 \leq 9^3$ implies $x \leq 9$, so 10 is not the smallest upper bound of A.

(b) Set $A = [0, 1]$ and $B = \{(x, y) \mid x^2 + y^2 = 1\}$. Then $A \times A = [0, 1] \times [0, 1]$ is a square, so $\mathbb{R}^2 - (A \times A)$ is the region outside this square, but B is the unit circle. □

2. Let $n \in \mathbb{Z}$. Show that 5 divides $6n$ if and only if 5 divides n. You may use basic properties of even and odd integers (i.e. what happens when you multiply two odd integers together, two even integers together, or an odd with an even), but no other properties of relatively prime integers. For instance, saying something along the lines of "if $6n$ is divisible by 5, then n is divisible by 5 since 6 and 5 are relatively prime" is not enough.

Proof 1. Suppose 5 divides n. Then there exists $k \in \mathbb{Z}$ such that $n = 5k$. Thus $6n = 30k = 5(6k)$, so 5 divides $6n$.

To prove the converse, we prove its contrapositive: if 5 does not divide n, then 5 does not divide $6n$. If 5 does not divide n, then we can write n as $n = 5k + r$ for some $r = 1, 2, 3, 4$. Then

$$6n = 6(5k + r) = 5(6k) + 6r.$$

If $r = 1$, this gives $6n = 5(6k+1)+1$, which is not divisible by 5; if $r = 2$, this gives $6n = 5(6k+2)+2$, which is not divisible by 5; if $r = 3$, we have $6n = 5(6k + 3) + 3$, which is not divisible by 5; and if $r = 4$, we get $6n = 5(6k + 4) + 4$, which is not divisible by 5. Hence in any case 5 does not divide $6n$ as claimed. □

Proof 2. The proof of the backwards direction is the same as above.

Suppose that 5 divides $6n$. Then $6n = 5k$ for some $k \in \mathbb{Z}$. If k were odd, then $5k = 6n$ would be odd, which $6n$ is even. Hence k is even so $k = 2m$ for some $m \in \mathbb{Z}$. Then

$$6n = 5(2k), \text{ so } 3n = 5k.$$

Now, if k is not divisible by 3, then either $k = 3\ell + 1$ or $k = 3\ell + 2$ for some $\ell \in \mathbb{Z}$. In these cases we get

$$3n = 5k = 15\ell + 5 = 3(5\ell + 1) + 2 \text{ or } 3n = 15\ell + 10 = 3(5\ell + 3) + 1,$$

which is not possible since neither $3(5\ell + 1) + 2$ nor $3(5\ell + 3) + 1$ are divisible by 3. Hence k must be divisible by 3, so $k = 3m$ for some $m \in \mathbb{Z}$. Then

$$3n = 5(3m), \text{ so } n = 5m$$

and hence 5 divides m. □

3. Let A and B be sets. Show that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

Proof. Let $x \in (A \cup B) - (A \cap B)$. Then $x \in A \cup B$ and $x \notin A \cap B$. Since $x \in A \cup B$, $x \in A$ or $x \in B$; without loss of generality we assume $x \in A$. But since $x \in A$ and $x \notin A \cap B$, we must have $x \notin B$. Hence $x \in A - B$, so $x \in (A - B) \cup (B - A)$. Thus

$$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A).$$

Conversely, let $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$, so without loss of generality we assume $x \in A - B$. Then $x \in A$ and $x \notin B$. Since $x \in A$, $x \in A \cup B$, and since $x \notin B$, $x \notin A \cap B$. Thus $x \in (A \cup B) - (A \cap B)$, so

$$(A \cup B) - (A \cap B) \supseteq (A - B) \cup (B - A).$$

Therefore we have equality as claimed. $\square$

4. At no point in either part below can you take some kind of limit. You must find a different way.

(a) Show that if x satisfies $6 - r^2 < x$ for all $r > 0$, then $6 \leq x$.

(b) Determine the following intersection and prove that your answer is correct.

$$\bigcap_{r>0} (6 - r^2, 9)$$

To be clear, $(6 - r^2, 9)$ denotes the interval $\left\{x \in \mathbb{R} \;\middle|\; 6 - r^2 < x < 9\right\}$ and the intersection is taken as r ranges among all positive real numbers.

Proof. (a) We prove the contrapositive: if $6 > x$, then there exists $r > 0$ such that $6 - r^2 \geq x$. If $6 > x$, then $6 - x > 0$ so $r = \frac{1}{2}\sqrt{6 - x}$ is defined and positive. Then

$$6 - x > \frac{1}{4}(6 - x) = r^2, \text{ so } 6 - r^2 > x,$$

which proves the contrapositive.

(b) We claim that this intersection equals $[6, 9)$. First, let $x \in \bigcap_{r>0}(6 - r^2, 9)$. Then $x \in (6 - r^2, 9)$ for all $r > 0$, so

$$6 - r^2 < x < 9 \text{ for all } r > 0.$$

By part (a), the inequalities on the left imply that $6 \leq x$, and since $x < 9$ as well we get $x \in [6, 9)$. Thus

$$\bigcap_{r>0} (6 - r^2, 9) \subseteq [6, 9).$$

Conversely let $x \in [6, 9)$, so that $6 \leq x < 9$. For any $r > 0$, $6 - r^2 < 6$ so:

$$6 - r^2 < 6 \leq x < 0 \text{ for all } r > 0.$$

Hence $x \in (6 - r^2, 9)$ for all $r > 0$, so $x \in \bigcap_{r>0}(6 - r^2, 9)$. Thus

$$\bigcap_{r>0} (6 - r^2, 9) \supseteq [6, 9),$$

so equality holds as claimed. $\square$

5. Let $A := \left\{3 - \frac{2}{\sqrt{n}} \;\middle|\; n \in \mathbb{N}\right\}$. Determine, with proof, the supremum of A. It would probably be easiest to use the fact that an upper bound b of A is the supremum of A if and only if for all $\epsilon > 0$, there exists $x \in A$ such that $b - \epsilon < x$, which you can use without justification.

Proof. We claim that $\sup A = 3$. First, for any $n \in \mathbb{N}$ we have

$$3 - \frac{2}{\sqrt{n}} < 3$$

since $\frac{2}{\sqrt{n}} > 0$, so 3 is an upper bound of A. Now, let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that

$$N > \frac{4}{\epsilon^2},$$

which exists by the Archimedean Property of $\mathbb{R}$. Then

$$\epsilon^2 > \frac{4}{N}, \text{ so } \epsilon > \frac{2}{\sqrt{N}}.$$

Thus

$$3 - \epsilon < 3 - \frac{2}{\sqrt{N}},$$

so $3 - \frac{2}{\sqrt{N}}$ is an element of A which is larger than $3 - \epsilon$, so $3 - \epsilon$ is not an upper bound of A. Hence nothing smaller than 3 is an upper bound of A, so 3 is indeed the smallest upper bound as claimed. $\square$