Math 300: Midterm 2 Solutions
Northwestern University, Spring 2017

1. Give an example of each of the following with brief justification.
(a) A function f and sets X, Y such that $f(X \cap Y) \neq f(X) \cap f(Y)$.
(b) A surjective function $f : \mathbb{Z} \to \mathbb{N}$ which is not invertible.

Solution. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$, $X = [-1, 0]$, and $Y = [0, 1]$. Then $X \cap Y = \{0\}$ so $f(X \cap Y) = \{0\}$, but $f(X) = [0, 1] = f(Y)$, so $f(X) \cap f(Y) = [0, 1]$.

(b) The function $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(x) = |x| + 1$ is surjective, since for any $n \in \mathbb{N}$ we have $f(n - 1) = n$, but not injective since $f(1) = f(-1)$. Hence f is not bijective, so it is not invertible. □

2. Suppose $x_1 > 1$ and define the numbers x_n recursively by

$$x_{n+1} = \frac{1 + x_n}{2} \text{ for } n \geq 1.$$

Show that $x_n > 1$ and $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.

Proof. We proceed by induction on n. First,

$$x_1 > 1 \text{ and } x_2 = \frac{1 + x_1}{2} < \frac{x_1 + x_1}{2} = x_1,$$

so the claimed inequalities hold in the $n = 1$ base case. Now suppose that $x_n > 1$ and $x_n \geq x_{n+1}$ for some n. Then

$$x_{n+1} = \frac{1 + x_n}{2} > \frac{1 + 1}{2} = 1$$

and

$$x_{n+2} = \frac{1 + x_{n+1}}{2} \leq \frac{1 + x_n}{2} = x_{n+1}.$$

Hence $x_n > 1$ and $x_n \geq x_{n+1}$ implies $x_{n+1} > 1$ and $x_{n+1} \geq x_{n+2}$, so by induction we conclude that $x_n > 1$ and $x_n \geq x_{n+1}$ hold for all n. □

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by

$$f(n) = \begin{cases} n + 2 & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd.} \end{cases}$$

Show that the image under f of the set of odd integers is the same as the image of the set of multiples of 4.

Proof. Let O denote the set of odd integers and M the set of multiples of 4. Let $x \in f(O)$. Then there exists $2k+1 \in O$ such that $f(2k+1) = x$, which means that $f(2k+1) = 2(2k+1) = 4k+2 = x$. But then

$$f(4k) = 4k + 2 = x,$$

so there exists $4k \in M$ such that $f(4k) = x$. Hence $x \in f(M)$, so $f(O) \subseteq f(M)$.

Now let $x \in f(M)$. Then there exists $4k \in M$ such that $f(4k) = x$, which means that $f(4k) = 4k + 2 = x$. But then

$$f(2k + 1) = 2(2k + 1) = 4k + 2 = x,$$

so $x \in f(O)$ since $2k + 1$ is an element of O mapping to x. Thus $f(M) \subseteq f(O)$, so $f(O) = f(M)$ as claimed. □

4. Suppose $f : A \to B$ is a function. Show that $f^{-1}(f(X)) = X$ for all $X \subseteq A$ if and only if f is injective.

Proof. Suppose $f^{-1}(f(X)) = X$ for all $X \subseteq A$, and suppose $a, a' \in A$ satisfy $f(a) = f(a')$. Then $a' \in f^{-1}(f(\{a\}))$ since $f(a') \in f(\{a\}) = \{f(a)\}$. By our assumption $f^{-1}(f(\{a\})) = \{a\}$, so $a' \in \{a\}$. Hence we must have $a' = a$, so f is injective.

Conversely suppose f is injective and let $X \subseteq A$. If $x \in X$, then $f(x) \in f(X)$ by definition of image, so $x \in f^{-1}(f(X))$ by definition of preimage. Thus $X \subseteq f^{-1}(f(X))$. Now suppose $y \in f^{-1}(f(X))$. Then $f(y) \in f(X)$, so there exists $x \in X$ such that $f(x) = f(y)$. Since f is injective, we get $x = y$, so $y \in X$ as well. Thus $f^{-1}(f(X)) \subseteq X$, so $f^{-1}(f(X)) = X$ as claimed. □

5. Define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ by

$$(m, n) \sim (a, b) \text{ if } m + b = n + a.$$

Show that $\sim$ is an equivalence relation and find a bijection between the set of equivalence classes and $\mathbb{Z}$. Hint: How can you uniquely characterize equivalence classes using integers? As a start, determine which elements of $\mathbb{N} \times \mathbb{N}$ are in the equivalence class of $(1, 1)$, and which are in the equivalence class of $(1, 2)$.

Solution. For any $(m, n) \in \mathbb{N} \times \mathbb{N}$, $m + n = n + m$ so $(m, n) \sim (m, n)$ and hence $\sim$ is reflexive. If $(m, n) \sim (a, b)$, then $m + b = n + a$, so $a + n = b + m$ as well. Hence $(a, b) \sim (m, n)$, so $\sim$ is symmetric. Finally, suppose $(m, n) \sim (a, b)$ and $(a, b) \sim (p, q)$. Then $m+b = n+a$ and $a+q = b+p$, so:

$$m + q = (n + a - b) + (b + p - a) = n + p.$$

Hence $(m, n) \sim (p, q)$, so $\sim$ is transitive and is thus an equivalence relation.

Fix $(m, n) \in \mathbb{N}\times\mathbb{N}$. Then $(a, b) \sim (m, n)$ when $a+n = b+m$, or equivalently when $m-n = a-b$. Thus $[(m, n)]$ consists of all pairs of positive integers whose difference (first coordinate minus second) gives the same integer as (m, n). The idea is that an equivalence class is then fully characterized by this integer difference, so the function from the set of equivalence classes to $\mathbb{Z}$ defined by

$$[(m, n)] \mapsto m - n$$

should be a bijection. Note first that this function is well-defined since $[(a, b)] = [(m, n)]$ give the same output $a - b = m - n$. If $[(m, n)]$ and $[(a, b)]$ both give the same output $m - n = a - b$, then $m + b = n + a$ so $(m, n) \sim (a, b)$ and hence $[(m, n)] = [(a, b)]$, showing that this function is injective. In addition, for any $x \in \mathbb{Z}$, picking positive integers $m, n \in \mathbb{N}$ such that $m - n = x$ gives a class $[(m, n)]$ which is sent to x, so this function is surjective as required.

The point of this problem is that this gives a way to "construct" the set of integers from the set of natural numbers. We *define* an "integer" to be an equivalence class of $\mathbb{N} \times \mathbb{N}$ under this equivalence relation, where we interpret $[(m, n)]$ as thus "representing" the integer $m - n$. □