Math 300: Midterm 2 Solutions
Northwestern University, Spring 2018

1. Give an example of each of the following with brief justification.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ and sets $X, Y \subseteq \mathbb{R}$ such that $f(X \backslash Y) \neq f(X) \backslash f(Y)$.

(b) An injective function $f : (0, 2) \to (0, 2)$ which is not invertible.

Solution. (a) Take $f(x) = x^2$, $X = [-1, 0]$ and $Y = [0, 1]$. Then $X \backslash Y = [-1, 0)$, so $f(X \backslash Y) = (0, 1]$. But $f(X) = f(Y) = [0, 1]$, so $f(X) \backslash f(Y) = \emptyset$.

(b) The function $f(x) = \frac{x}{2}$ works. It is injective since $\frac{x}{2} = \frac{y}{2}$ implies $x = y$, but it is not surjective since no $x \in (0, 2)$ satisfies $f(x) = \frac{x}{2} = 1.5$. (Note that for $x \in (0, 2)$, $\frac{x}{2}$ is still in $(0, 2)$ so f indeed maps $(0, 2)$ into $(0, 2)$.) $\square$

2. For a complex number $z = a + ib$, where a and b are real numbers, the *complex conjugate* of z is the complex number $\overline{z} = a - ib$. Show that for any n complex numbers, where $n \geq 2$, the following equality holds:

$$\overline{z_1 z_2 \cdots z_n} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_n}.$$

Proof. Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where $a_1, b_1, a_2, b_2 \in \mathbb{R}$. Then

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1),$$

so

$$\overline{z_1 z_2} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1).$$

Now,

$$\overline{z_1}\,\overline{z_2} = (a_1 - ib_1)(a_2 - ib_2) = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1),$$

so we see that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, so the required identity holds for the base case of $n = 2$.

Suppose now that for some $n \geq 2$, the required identity holds for any n complex numbers. Let $z_1, \ldots, z_{n+1}$ be $n + 1$ complex numbers. Then

$$\overline{z_1 \cdots z_{n-1} z_n} = \overline{(z_1 \cdots z_{n-1}) z_n} = \overline{z_1 \cdots z_{n-1}}\,\overline{z_n}$$

by the base case, and

$$\overline{z_1 \cdots z_{n-1}} = \overline{z_1} \cdots \overline{z_{n-1}}$$

by the induction hypothesis. Putting it all together gives

$$\overline{z_1 \cdots z_{n+1}} = \overline{z_1} \cdots \overline{z_{n+1}},$$

so the required identity holds for $n + 1$ complex numbers. Hence by induction we conclude that it holds for any $n \geq 2$ complex numbers. $\square$

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by

$$f(n) = \begin{cases} 2n + 1 & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Show that the **image** of $2\mathbb{Z}$ is equal to the **inverse image** of $4\mathbb{Z}$. (Recall that $2\mathbb{Z}$ denotes the set of even integers and $4\mathbb{Z}$ the set of multiples of 4.)

Proof. Let $b \in f(2\mathbb{Z})$. Then there exists $2k \in 2\mathbb{Z}$, where $k \in \mathbb{Z}$, such that $f(2k) = b$. By the definition of f, this gives

$$2(2k) + 1 = b, \text{ so } b = 4k + 1.$$

Since b is then odd, we have $f(b) = b - 1 = (4k+1) - 1 = 4k \in 4\mathbb{Z}$. Hence $b \in f^{-1}(4\mathbb{Z})$, so we have that $f(2\mathbb{Z}) \subseteq f^{-1}(4\mathbb{Z})$.

Now suppose $y \in f^{-1}(4\mathbb{Z})$. Then $f(y) \in 4\mathbb{Z}$, so $f(y) = 4k$ for some $k \in \mathbb{Z}$. By the definition of f, in order for $f(y)$ to be even, y must be odd, so $4k = f(y) = y - 1$. Hence $y = 4k + 1$. Thus

$$f(2k) = 2(2k) + 1 = 4k + 1 = y,$$

so $y \in f(2\mathbb{Z})$. Hence $f^{-1}(4\mathbb{Z}) \subseteq f(2\mathbb{Z})$, so $f(2\mathbb{Z}) = f^{-1}(4\mathbb{Z})$ as claimed. □

4. Suppose $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f : A \to C$ is bijective. Show that g is injective if and only if f is surjective.

Proof. Suppose g is injective. Let $b \in B$. Then $g(b) \in C$, so since $g \circ f$ is surjective, there exists $a \in A$ such that $g(f(a)) = g(b)$. Since g is injective, $f(a) = b$, so we have found $a \in A$ which f sends to b. Since $b \in B$ was arbitrary, this shows that f is surjective.

Conversely suppose f is surjective. Suppose $b_1, b_2 \in B$ are such that $g(b_1) = g(b_2)$. Since f is surjective, there exist $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Then we have

$$g(f(a_1)) = g(f(a_2)), \text{ so } a_1 = a_2$$

since $g \circ f$ is injective. Applying f gives $f(a_1) = f(a_2)$, so $b_1 = b_2$. Hence g is injective. □

5. Let $\mathbb{R}^*$ denote the set of nonzero real numbers. Define a relation on $\mathbb{R}^* \times \mathbb{R}^*$ by saying

$$(x, y) \sim (a, b) \text{ if } xa > 0 \text{ and } yb > 0.$$

Show that $\sim$ is an equivalence relation, and show that there are only **four** distinct equivalence classes, which you should be able to describe explicitly. (So, the equivalence class of any point (x, y) will be equal to one of these four.)

Proof. Let $(x, y) \in \mathbb{R}^* \times \mathbb{R}^*$. Since x and y are nonzero, $xx > 0$ and $yy > 0$. Hence $(x, y) \sim (x, y)$, so $\sim$ is reflexive. Suppose $(x, y) \sim (a, b)$. Then $xa > 0$ and $yb > 0$. But this is the same as $ax > 0$ and $by > 0$, so $(a, b) \sim (x, y)$. Hence $\sim$ is symmetric.

Finally suppose $(x, y) \sim (a, b)$ and $(a, b) \sim (p, q)$. Then $xa > 0$, $yb > 0$, $ap > 0$, and $bq > 0$. Since $ap > 0$ and $a \neq 0$, $\frac{p}{a} > 0$. Thus

$$xp = (xa)\frac{p}{a} > 0$$

since the right is the product of two positive numbers. Similarly, since $bq > 0$ and $b \neq 0$, $\frac{q}{b} > 0$. So

$$yq = (yb)\frac{q}{b} > 0.$$

Thus $(x, y) \sim (p, q)$, so $\sim$ is transitive.

Now, $(x, y) \sim (1, 1)$ if and only if $x(1) > 0$ and $y(1) > 0$. Thus the equivalence class of $(1, 1)$ consists of all points whose coordinates are both positive, so $[(1, 1)]$ is the first quadrant of $\mathbb{R}^* \times \mathbb{R}^*$. Next, $(x, y) \sim (-1, 1)$ if and only if $x(-1) > 0$ and $y(1) > 0$, which says that $x < 0$ and $y > 0$. Thus $[(-1, 1)]$ consists of all points with negative x-coordinate and positive y-coordinate, so it is the second quadrant. Similarly, $(x, y) \sim (-1, 1)$ if and only if x, y are both negative, so $[(-1, 1)]$ is the third quadrant, and $(x, y) \sim (1, -1)$ if and only if x is positive and y is negative, so $[(1, -1)]$ is the fourth quadrant. Since these four equivalence classes already cover everything in $\mathbb{R}^* \times \mathbb{R}^*$, we conclude that they are the only distinct equivalence classes as claimed. □