Math 300: Midterm 2 Solutions
Northwestern University, Winter 2019

1. Give an example of each of the following with brief justification.

(a) An invertible function $f : \mathbb{Z} \to \mathbb{Z}$ which is not the identity function.

(b) A subset X of $\mathbb{R}$ such that $f^{-1}(f(X)) \neq X$ where $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = x^2$.

Solution. (a) The function $f(n) = n + 1$ is invertible since it has inverse $f^{-1}(n) = n - 1$, but is not the identity function.

(b) The subset $X = [0, 1]$ works. We have $f(X) = [0, 1]$ and $f^{-1}(f(X)) = [-1, 1]$, which contains more than $[0, 1]$. □

2. Suppose $x_1 < 2$ and define the numbers x_n for $n > 1$ by setting

$$x_{n+1} = \frac{4x_n + 2}{5} \text{ for } n \geq 1.$$

Show that 2 is an upper bound of the set $\{x_n \mid n \in \mathbb{N}\}$ containing these numbers.

Proof. By assumption, $x_1 < 2$. Suppose now that $x_n < 2$ for some n. Then:

$$x_{n+1} = \frac{4x_n + 2}{5} < \frac{4(2) + 2}{5} = \frac{10}{5} = 2.$$

Thus by induction we conclude that $x_n < 2$ for all $n \in \mathbb{N}$, so 2 is an upper bound of the given set containing the numbers x_n of this sequence. □

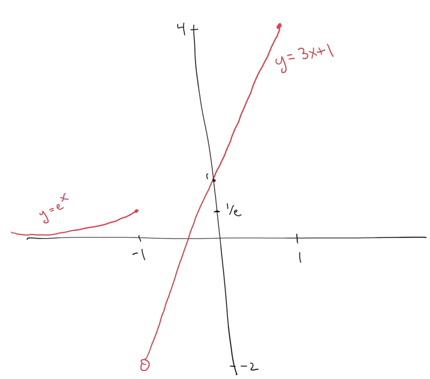
3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 3x + 1 & \text{if } x > -1 \\ e^x & \text{if } x \leq -1 \end{cases}$$

(a) Determine, with proof, the image $f([-2, 1])$ of the interval $[-2, 1]$ under f. (You can draw a picture of the graph of f to come up with a guess as to what the answer should be, but the picture alone does not constitute a proof.)

(b) Also, determine what the preimage $f^{-1}([1/3, 1])$ of the interval $[1/3, 1]$ is, but you do not have to prove that your answer is correct in this case.

Solution. As a reference, the graph of f looks like:



(a) We claim that $f([-2,1]) = (-2,4]$. First, suppose $y \in f([-2,1])$. Then there exists $x \in [-2,1]$ such that $f(x) = y$. If $-1 < x \leq 1$, then $y = f(x) = 3x+1$ which lies in $(-2,4]$ since $-1 < x \leq 1$:

$$-3 < 3x \leq 3, \text{ so } -2 < 3x+1 \leq 4.$$

If $-2 \leq x \leq -1$, then $y = f(x) = e^x$ which also lies in $(-2,4]$:

$$-2 < e^{-2} \leq e^x \leq e^{-1} \leq 4.$$

Thus either way we have $y \in (-2,4]$, so $f([-2,1]) \subseteq (-2,4]$.

Conversely suppose $y \in (-2,4]$. Then $-3 < y-1 \leq 3$ so

$$-1 < \frac{y-1}{3} \leq 1.$$

Since $x := \frac{y-1}{3} \in (-1,1] \subseteq [-2,1]$ satisfies $f(x) = 3x+1 = 3\left(\frac{y-1}{3}\right)+1 = y$, we conclude that $y \in f((-2,1])$. Hence $f([-2,1]) \supseteq (-2,4]$, so we have equality as claimed.

(b) We want elements x such that $f(x) \in [1/3,1]$. First, for $x \leq -1$, this requires that

$$\frac{1}{3} \leq e^x \leq 1, \text{ so } \ln\frac{1}{3} \leq x \leq \ln 1 = 0,$$

and in fact the right side can be strengthened to $x \leq -1$ since in this case we are considering $x \leq -1$. Thus numbers in $[\frac{1}{3}, -1]$ are in the preimage of $[1/3,1]$. For $x > -1$, the requirement is that

$$\frac{1}{3} \leq 3x+1 \leq 1, \text{ so } -\frac{2}{9} \leq x \leq 0.$$

These numbers should also be in the preimage, so

$$f^{-1}([1/3,1]) = [-\tfrac{2}{9}, 0] \cup [\ln\tfrac{1}{3}, -1].$$

□

4. Suppose $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f : A \to C$ is injective. Show that f is injective. (We did this in class, but of course you cannot simply quote this exact result from class.)

Solution. Suppose $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$. Then $g(f(a_1)) = g(f(a_2))$, or in other words $(g \circ f)(a_1) = (g \circ f)(a_2)$. Since $g \circ f$ is injective, we conclude that $a_1 = a_2$, and hence that f is injective as claimed. □

5. Define a relation $\sim$ on $\mathbb{R}^2$ by setting

$$(x,y) \sim (a,b) \text{ if } y-b = 3(x-a)$$

(a) Show that $\sim$ is an equivalence relation.

(b) For a fixed point $(x,y) \in \mathbb{R}^2$ find the element in the equivalence class $[(x,y)]$ of (x,y) which is on the x-axis. (You can take for granted the fact that each equivalence class intersects the x-axis in exactly one point.)

Solution. (a) For $(x, y) \in \mathbb{R}^2$, we have

$$y - y = 3(x - x),$$

so $(x, y) \sim (x, y)$ and hence $\sim$ is reflexive. Suppose $(x, y) \sim (a, b)$. Then

$$y - b = 3(x - a), \text{ so } -(b - y) = -3(a - x).$$

This gives $b - y = 3(a - x)$, so $(a, b) \sim (x, y)$ and hence $\sim$ is symmetric. Finally, suppose $(x, y) \sim (a, b)$ and $(a, b) \sim (p, q)$. Then

$$y - b = 3(x - a) \quad \text{and} \quad b - q = 3(a - p).$$

Adding these together gives

$$y - q = 3(x - a) + 3(a - p) = 3(x - p),$$

so $(x, y) \sim (p, q)$ and thus $\sim$ is transitive. Hence $\sim$ is an equivalence relation as claimed.

(b) Points in $[(x, y)]$ are those points (a, b) which satisfy

$$y - b = 3(x - a).$$

In order for (a, b) to be on the x-axis we need $b = 0$, in which case the condition above becomes

$$y = 3(x - a).$$

Thus $a = \frac{3x-y}{3}$. Hence the point in the equivalence class $[(x, y)]$ which is on the x-axis is the point $(\frac{3x-y}{3}, 0)$. $\square$