Math 300: Midterm 2

Northwestern University, Spring 2017

Name: ______________________________

1. (10 points) Give an example of each of the following with brief justification.

(a) A function f and sets X, Y such that $f(X \cap Y) \neq f(X) \cap f(Y)$.

(b) A surjective function $f : \mathbb{Z} \to \mathbb{N}$ which is not invertible.

Problem	Score
1	
2	
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Total	

2. (10 points) Suppose $x_1 > 1$ and define the numbers x_n recursively by

$$x_{n+1} = \frac{1 + x_n}{2} \text{ for } n \geq 1.$$

Show that $x_n > 1$ and $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.

3. (10 points) Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by

$$f(n) = \begin{cases} n+2 & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd.} \end{cases}$$

Show that the image under f of the set of odd integers is the same as the image of the set of multiples of 4.

4. (10 points) Suppose $f : A \to B$ is a function. Show that $f^{-1}(f(X)) = X$ for all $X \subseteq A$ if and only if f is injective.

5. (10 points) Define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ by

$$(m, n) \sim (a, b) \text{ if } m + b = n + a.$$

Show that $\sim$ is an equivalence relation and find a bijection between the set of equivalence classes and $\mathbb{Z}$. Hint: How can you uniquely characterize equivalence classes using integers? As a start, determine which elements of $\mathbb{N} \times \mathbb{N}$ are in the equivalence class of $(1, 1)$, and which are in the equivalence class of $(1, 2)$.