NOTES ON FUNCTIONS

These notes will cover some terminology regarding functions not included in Solow's book. You should read Appendix A.2 in the book before reading these notes.

**Definition 1.** We say that two functions \( f \) and \( g \) are equal if they have the same domain and codomain, and \( f(a) = g(a) \) for all \( a \) in the domain.

Note that we require the functions to have the same domain and codomain for them to be equal. For example, the functions \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{Z} \to \mathbb{Z} \) given by \( f(x) = x^2 \) and \( g(x) = x^2 \) are defined by the same formula, but they are not equal since they have different domains and codomains. The function \( h : \mathbb{R} \to \{ x \in \mathbb{R} \mid x \geq 0 \} \) given by \( h(x) = x^2 \) again is defined by the same formula as \( f \), and now has the same domain as \( f \), but since they have different codomains, they are not equal.

**Definition 2.** The *identity function* on a set \( A \), denoted by \( \text{id}_A \), is the function from \( A \) to itself such that \( \text{id}_A(a) = a \) for all \( a \in A \).

**Definition 3.** If \( f : A \to B \) and \( g : B \to C \) are functions, we define their *composition*, denoted by \( g \circ f \), to be the function \( g \circ f : A \to C \) defined by \( (g \circ f)(a) = g(f(a)) \).

**Definition 4.** A function \( f : A \to B \) is called *bijective* if it is both injective and surjective.

Again, we have to be careful about the domain and codomain on which a function is defined. Consider a function given by the formula \( f(x) = x^2 \). It makes no sense to say this is injective, surjective, nor bijective without specifying what domain and codomain we are considering. For example, as a function from \( \mathbb{R} \) to \( \mathbb{R} \), \( f \) is neither injective nor surjective; as a function from \( \mathbb{R} \) to \( \{ x \in \mathbb{R} \mid x \geq 0 \} \), it is surjective but not injective; and as a function from \( \{ x \in \mathbb{R} \mid x \geq 0 \} \) to itself, it is bijective.

**Definition 5.** Let \( f : A \to B \) be a function. We say that \( f \) is *invertible* if there is a function \( g : B \to A \) such that \( g \circ f = \text{id}_A \) and \( f \circ g = \text{id}_B \). In this case we call \( g \) the inverse of \( f \) and denote it by \( f^{-1} \).

This notion also depends on the domain and codomain; the function \( h(x) = x^2 \) is invertible as a function from the set of positive real numbers to itself (its inverse in this case is the square root function), but it is not invertible as a function from \( \mathbb{R} \) to \( \mathbb{R} \). The following theorem shows why:

**Theorem 1.** A function is invertible if and only if it is bijective.

**Proof.** Suppose that the function \( f : A \to B \) is invertible and let \( f^{-1} \) be its inverse. First we show that \( f \) is injective. To this end, suppose \( a, b \in A \) are such that \( f(a) = f(b) \). Then we can apply \( f^{-1} \) to both sides to get

\[
(f^{-1} \circ f)(a) = (f^{-1} \circ f)(b).
\]
By the definition of inverses, both compositions above are the identity on \( A \), so the above equality becomes \( a = b \), showing that \( f \) is injective. To show that \( f \) is surjective, let \( b \in B \) and let \( a = f^{-1}(b) \). Then
\[
f(a) = f(f^{-1}(b)) = (f \circ f^{-1})(b) = \text{id}_B(b) = b
\]
as required. We conclude that \( f \) is bijective.

Conversely, suppose that \( f \) is bijective. We must construct an inverse. Let \( b \in B \). Since \( f \) is surjective, there exists \( a_b \in A \) so that \( f(a_b) = b \), and since \( f \) is injective, there is only one such \( a_b \). We define a function \( g : B \to A \) by setting \( g(b) = a_b \). It is then easy to check (which you should!) that \( g \) satisfies the defining properties of the inverse of \( f \), so \( f \) is invertible. \( \square \)

Next we look at certain sets which are associated to functions.

**Definition 6.** Let \( f : A \to B \) be a function. The *image* of a subset \( U \subseteq A \) under \( f \), denoted by \( f(U) \), is the set of all elements \( b \) of \( B \) for which there exists \( a \in U \) so that \( f(a) = b \); in set notation this means
\[
f(A) = \{ b \in B \mid \exists a \in U \text{ such that } f(a) = b \} = \{ f(a) \in B \mid a \in U \}.
\]
In particular, the image of \( A \) is called the image (or *range*) of \( f \); in addition to the notation \( f(A) \), the image of \( f \) is also denoted by \( \text{im}(f) \).

If \( C \subseteq B \), the *preimage* (or *inverse image*) of \( C \), denoted by \( f^{-1}(C) \), is the set of all \( a \in A \) so that \( f(a) \in C \); in set notation this means
\[
f^{-1}(C) = \{ a \in A \mid f(a) \in C \}.
\]
For \( b \in B \), the notation \( f^{-1}(b) \) is commonly used for \( f^{-1}(\{b\}) \).

So, the image of a subset \( U \) of \( A \) under a function \( f : A \to B \) is the set of all things in \( B \) which you can possibly get by applying \( f \) to elements of \( U \), and the preimage of a set \( C \) is the set of all things in \( A \) which are sent into \( C \) by \( f \). The preimage of a single element \( b \) of \( C \), i.e. the set of all things in \( A \) that map to \( b \), is commonly called the *fiber* of \( f \) above \( b \); we won’t be using this terminology in this class, and indeed you probably won’t see it again unless you take a geometry or topology course later on. If you really do want to know why we use the word “fiber”, I’d be happy to tell you in office hours!

**Exercises**

1. Let \( f : A \to B \) and \( g : B \to C \) be functions. Prove that if the composition \( g \circ f \) is injective, then \( f \) is injective. Prove that if the composition \( g \circ f \) is surjective, then \( g \) is surjective.

2. Let \( f : A \to B \) be a function, and let \( U \subseteq A \). Prove that \( U \subseteq f^{-1}(f(U)) \). Are these sets necessarily equal? Why or why not? If not, what are some conditions on \( f \) under which they will be equal?

3. Let \( g : C \to D \) be a function, and let \( V \subseteq D \). Prove that \( f(f^{-1}(V)) \subseteq V \). Are these sets necessarily equal? Why or why not? If not, what are some conditions on \( g \) under which they will be equal?
4. Let \( f : A \to B \) be a function, and let \( C, D \subseteq A \). Must it be true that \( f(C \cap D) = f(C) \cap f(D) \)? Must it be true that \( f(C \cup D) = f(C) \cup f(D) \)? For each of these, either prove the given equality or give an example in which it fails.

5. There is a one-to-one correspondence between functions from \( A \) to \( B \) and relations \( \Gamma \) between \( A \) and \( B \) satisfying the property that if \( (a, b) \in \Gamma \) and \( (a, c) \in \Gamma \), then \( b = c \). Explicitly, a function \( f : A \to B \) corresponds to the relation
\[
\gamma = \{(a, f(a)) \in A \times B\},
\]
called the graph of the function, and conversely a relation
\[
\gamma = \{(a, b) \in A \times B\}
\]
satisfying the above mentioned property corresponds to the function \( f \) defined by \( f(a) = b \). Indeed, this is essentially the definition of a function given in the book.

Under this correspondence between functions and relations, if the graph of a function is actually an equivalence relation, what can be said about the function in question?