NOTES ON REAL NUMBERS

In these notes we will construct the set of real numbers. Why would we want to do this you may ask? Well, mathematicians want mathematics to be based on a solid foundation, such as set theory. Adopting this point of view then requires us to give precise definitions of the different things we use in math; for example, we've seen how to give precise definitions of "ordered pair" and "function" in terms of set theory. So somewhere along the way, someone had to give a precise definition of the real numbers, built up from set theory as well. We will give one such construction here. At the end we will have a precise definition of $\mathbb{R}$, but just as we did for ordered pairs and functions, we will then forget about the precise construction and just use real numbers as we are used to them. Still, this is a good thing to go through since it gives some down to earth applications of some of the tools we have been studying, especially equivalence relations, to create some interesting mathematics.

As a word of caution, this material is not easy and may seem very confusing at first. Because of this, this material is not something that quizzes nor the rest of the course will focus on. However, this material does give a good glimpse into the kinds of things that modern mathematicians do and the kinds of things that modern mathematical research is concerned with, although this specific material was first developed towards the end of the 19th century and so is not exactly modern. You can definitely expect to see the types of constructions and ideas described here in later upper division courses, and possibly in your future mathematical careers.

The only assumptions we will make are that the set of natural numbers $\mathbb{N}$ exists and that we know how to add and multiply natural numbers. In fact, that this set and these operations exist follows from commonly accepted axioms of set theory (in particular, the "axiom of infinity"), but we will just take their existence for granted. It is truly amazing that such a simple set of axioms gives rise to the rich complexity of the real numbers!

We will first construct the set of integers. We want to think of $(a, b) \in \mathbb{N} \times \mathbb{N}$ as representing the integer $a - b$. The first problem with this is that there are many ways of representing an integer as a difference $a - b$; for example, $(2, 7)$, $(1, 6)$, and $(95, 100)$ all give the same integer, namely -5. So, to deal with this we will define an equivalence relation on $\mathbb{N} \times \mathbb{N}$ in a way so that equivalent pairs correspond to the same integer. Then, even though different pairs give the same integer, there will in fact be a unique equivalence class which gives a certain integer.

Now, since we want two pairs (a, b) and (c, d) to be equivalent when the integers they represent, $a - b$ and $c - d$ respectively, are the same, at first we might try to define the equivalence relation we need by saying that

$$(a, b)R(c, d) \text{ if } a - b = c - d.$$

But now we come across another problem; namely that the operation of subtraction already depends on the existence of the integers, so that is no good if we are trying to construct the integers. Instead we note that $a-b=c-d$ if and only if $a+d=b+c$, which is good since we already know how to add natural numbers. Hence we can use this second equality as the defining property of our equivalence relation. Thus we define an equivalence relation on $\mathbb{N}\times\mathbb{N}$ by saying

$$(a,b)\sim(c,d) \text{ if } a+d=b+c,$$

and finally we make the definition:

Definition 1. The set of integers, denoted by $\mathbb{Z}$, is defined to be $\mathbb{N}\times\mathbb{N}/\sim$.

Again, we think of the equivalence class $[(a,b)]$ as representing the integer $a-b$; the fact that the relation defined above is in fact an equivalence relation guarantees that anything in the equivalence class of (a,b) represents the same integer. So for example, the integer -5 can be represented by the equivalent pairs $(2,7)$, $(1,6)$, and $(95,100)$. The integer 0 can be represented by any pair of the form (a,a).

Now, using this definition of the integers as the equivalence classes of a certain equivalence relation, how would we define the "sum" of two integers? Well, the sum of two "integers" $[(a,b)]$ and $[(c,d)]$ should be another equivalence class. Since $[(a,b)]$ and $[(c,d)]$ represent $a-b$ and $c-d$ respectively, it makes sense to define their sum as the equivalence class representing the integer $(a-b)+(c-d)$. Rewriting this as $(a+c)-(b+d)$, we see that this integer is represented by the pair $(a+c,b+d)$. Thus we define the sum of two equivalence classes $[(a,b)]$ and $[(c,d)]$ as

$$[(a,b)]+[(c,d)]:=[(a+c,b+d)].$$

Now we have to know that this sum is "well-defined". In other words, we know that any pair (a',b') equivalent to (a,b) gives the same equivalence class as (a,b); that is $[(a',b')]=[(a,b)]$ if $(a',b')\sim(a,b)$. So we could have also used (a',b') when defining the above sum:

$$[(a',b')]+[(c,d)]=[(a'+c,b'+d)].$$

This sum should not depend on whether we use (a,b) or an equivalent pair (a',b'); i.e. it should be "well-defined". In this case, this means that even if we use (a',b') instead of (a,b), the equivalence classes we get as the sum

$$[(a+c,b+c)] \text{ and } [(a'+c,b'+c)]$$

should in fact be equal. Remembering the condition under which two equivalence classes are equal, this requires that $(a+c,b+c)$ and $(a'+c,b'+c)$ be equivalent pairs, which now prove.

Proposition 1. *Let $\sim$ be the above equivalence relation on $\mathbb{N}\times\mathbb{N}$. Then if $(a,b)\sim(a',b')$,*

$$(a+c,b+d)\sim(a'+c,b'+d)$$

for any $(c,d)\in\mathbb{N}\times\mathbb{N}$.

Proof. Suppose that $(a,b)\sim(a',b')$ and let $(c,d)\in\mathbb{N}\times\mathbb{N}$. By the definition of the equivalence relation $\sim$, we must show that

$$(a+c)+(b'+d)=(b+d)+(a'+c).$$

Since $(a, b) \sim (a', b')$, we know that

$$a + b' = b + a'.$$

Thus adding $c + d$ to both sides of this gives

$$a + b' + c + d = b + a' + c + d,$$

which is exactly the first equality which we wanted to establish. Hence $(a + c, b + d) \sim (a' + c, b' + d)$. $\square$

This result tells us even if we use different pairs to represent an equivalence class when computing the sum of two equivalence classes, the resulting equivalence class we get as the sums will in fact be equal. Hence the given definition of the sum of two integers, thought of as equivalence classes, is well-defined. Also, we define the "product" of two equivalence classes as

$$[(a, b)] \cdot [(c, d)] = [(ac + bd, ad + bc)].$$

Thinking of the equivalence classes $[(a, b)]$ and $[(c, d)]$ as representing the integers $a - b$ and $c - d$ respectively, this above definition should correspond to taking the product of $a - b$ and $c - d$. I encourage you to think about this on your own and see why this is the correct notion of multiplication; in the exercises you will show that this is well-defined.

Now we construct the set of rational numbers. First, we denote the set of nonzero integers by $\mathbb{Z}^*$. Define an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^*$ by saying

$$(a, b) \sim (c, d) \text{ if } ad = bc.$$

Similar to the above construction, we want to say that (a, b) represents the rational number $\frac{a}{b}$. First note that since the second component of a pair (a, b) is in $\mathbb{Z}^*$, it is nonzero so it makes sense for it to be the denominator of a rational number. The problem here is that the operation of division already depends on the existence of $\mathbb{Q}$, so we cannot use it if trying to define the rational numbers. Instead we realize that $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$, and we know how to multiply integers; this is the motivation for the equivalence relation defined above. We then make the definition:

Definition 2. The set of rational numbers, denoted by $\mathbb{Q}$, is defined to be $\mathbb{Z} \times \mathbb{Z}^* / \sim$.

An equivalence class $[(a, b)]$ represents the rational number $\frac{a}{b}$, and again anything in this equivalence class represents the same rational number since the relation is an equivalence relation. For example, the rational number $\frac{1}{2}$ can be represented by the pairs $(1, 2)$, $(4, 8)$, or $(-3, -6)$. We define the product of two rational numbers $[(a, b)]$ and $[(c, d)]$ by

$$[(a, b)] \cdot [(c, d)] = [(ac, bd)],$$

which can be shown to be well-defined.

Finally we come to the construction of the real numbers, which is unfortunately not as simple as defining an equivalence relation on $\mathbb{Q} \times \mathbb{Q}$ or some similar set. Part of the reason for this is that any such construction will only give a countable set, yet we know that the set of real numbers is uncountable. Instead, we will use a construction based on the idea of thinking of the real numbers as "filling in the gaps" of $\mathbb{Q}$.

Definition 3. A *Dedekind cut* of $\mathbb{Q}$ is a pair (A, B) of nonempty subsets of $\mathbb{Q}$ satisfying the following properties:

(1) A and B are disjoint and their union is $\mathbb{Q}$,
(2) If $a \in A$, then every $r \in \mathbb{Q}$ such that $r < a$ is also in A,
(3) If $b \in B$, then every $r \in \mathbb{Q}$ such that $b < r$ is also in B, and
(4) A contains no maximum element.

A Dedekind cut gives a division of the set of rational numbers into two sets A and B, with all rationals to the "left of the division point" being in A and all rationals to the "right of the division point" being in B. The "division point" is what we should think of as representing a real number.

For example, let

$$A = \{x \in \mathbb{Q} \mid x^2 < 2 \text{ or } x < 0\} \text{ and } B = \{x \in \mathbb{Q} \mid x^2 > 2 \text{ and } x > 0\}.$$

Then the pair (A, B) forms a Dedekind cut, as can easily be checked. We want to think of this particular Dedekind cut as representing the real number $\sqrt{2}$, which is the "division point" between those rational numbers whose square is less than 2 or which are negative, and those which are positive and whose square is greater than 2. Instead you might want to try and define this division using the sets

$$A = \{x \in \mathbb{Q} \mid x < \sqrt{2}\} \text{ and } B = \{x \in \mathbb{Q} \mid x \geq \sqrt{2}\},$$

but the problem with this is that we do not yet know what $\sqrt{2}$ means. If you think about it for a while (or in fact, a long while), you can convince yourself that every real number can be represented by such a Dedekind cut, again by thinking of the given real number as the "division point" of the cut, and that such a Dedekind cut describes exactly one real number. Thus we make the definition:

Definition 4. The set of real numbers, denoted by $\mathbb{R}$, is the set of all Dedekind cuts of $\mathbb{Q}$.

Note in particular that a rational number r, thought of as a real number, is represented by the Dedekind cut given by

$$\{x \in \mathbb{Q} \mid x < r\} \text{ and } \{x \in \mathbb{Q} \mid x \geq r\}.$$

This is not an easy definition to wrap your head around. Indeed, to even show that each real number does correspond to such a Dedekind cut requires quite a bit of Math 104 material. In Math 104 you will learn about what are called "Cauchy sequences", which can be used to give a somewhat more intuitive construction of $\mathbb{R}$. For now, you'll just have to trust that Dedekind cuts do indeed give a precise definition of $\mathbb{R}$.

From here, to fully develop the real numbers we would need to define addition, subtraction, multiplication, and other such operations on Dedekind cuts in a way which is consistent with how we perform these operations on $\mathbb{R}$. We should also show that $\mathbb{R}$ defined as the set of Dedekind cuts of $\mathbb{Q}$ satisfies all the properties we have come to expect of the real numbers; for example, we would have to show that the set of Dedekind cuts satisfies the Infimum Property of $\mathbb{R}$ given below and in Appendix D.1. All of this can be done but can be somewhat tedious. You can check other resources (such as Wikipedia!) for further details.

Additional Properties of $\mathbb{R}$

You should read Appendix D.1 in the book before reading this section. Recall the *Infimum Property of* $\mathbb{R}$: Any nonempty set of real numbers which is bounded below has an infimum (i.e. greatest lower bound). Here is another important definition and two further properties of $\mathbb{R}$:

Definition 5. An upper bound s of a set of real numbers A is called the *supremum* of A (or *least upper bound* of A) if $s \leq b$ for any other upper bound b of A

Supremum Property of $\mathbb{R}$: Any nonempty set of real numbers which is bounded above has a supremum.
Archimedean Property of $\mathbb{R}$: For any positive real number x, there exists a natural number N so that $x < N$.

Exercises

1. Show that the relation on $\mathbb{N} \times \mathbb{N}$ given by

$$(a, b) \sim (c, d) \text{ if } a + d = b + c$$

is indeed an equivalence relation.

2. Show that the relation on $\mathbb{Z} \times \mathbb{Z}^*$ given by

$$(a, b) \sim (c, d) \text{ if } ad = bc$$

is indeed an equivalence relation.

3. Show that the definition of multiplication on $\mathbb{Z}$ given in the notes is well-defined; that is, show that if $(a, b) \sim (a', b')$, then

$$(a, b) \cdot (c, d) \sim (a', b') \cdot (c, d)$$

for any "integer" (c, d).

4. Give a (well-defined) definition for the sum of two rational numbers $[(a, b)]$ and $[(c, d)]$. Again, being well-defined means that your definition of sum should not depend on which "rational" we use to represent a given equivalence class; more precisely, if $(a, b) \sim (a', b')$, your definition of the sum of these two "rational numbers" should have the property that

$$(a, b) + (c, d) \sim (a', b') + (c, d)$$

for any "rational" (c, d).

5. Give a guess for how you might try to define the sum of two Dedekind cuts.

6. Show that the Infimum Property of $\mathbb{R}$ and the Supremum Property of $\mathbb{R}$ are equivalent to each other.

7. Prove that the Archimedean Property of $\mathbb{R}$ is equivalent to the following: For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$.