NOTES ON RELATIONS

These notes introduce the notion of a relation and, more importantly, the notion of an equivalence relation.

Definition 1. A *relation* R between two sets A and B is a subset of the Cartesian product $A \times B$. We say that $a \in A$ and $b \in B$ are related if $(a, b) \in R$ and denote this by writing aRb. By a relation on a set A we mean a relation between A and itself.

This definition may seem to be of little use, since we are just defining a relation as a subset of a Cartesian product with no additional requirements. The point is that it gives us a set-theoretic and precise way of defining what it means for two objects to be related (in whatever sense we are working with) to each other.

Here is a simple example. Let A be the set of all people in the world and define a relation on A by

$$R := \{(a, b) \in A \times A \mid a \text{ is the father of } b\}.$$

This set completely encodes the relation between father and child in a precise way, which is what we would want if we wanted to do some mathematics with this (which we do not).

The most important types of relations are those which are known as *equivalence relations*:

Definition 2. A relation on a set A is called an *equivalence relation* if it satisfies the following properties:

- The relation is *reflexive*, meaning that aRa for all $a \in A$.
- The relation is *symmetric*, meaning that aRb implies bRa.
- The relation is *transitive*, meaning that if aRb and bRc, then aRc.

Equivalence relations are commonly denoted by $\sim$, so instead of writing aRb we write $a \sim b$ and say that a and b are equivalent.

What are some examples of equivalence relations? The example of the relation between father and child given above is not since it is not reflexive (you cannot be your own father), symmetric (you cannot be your father's father), nor transitive (your father cannot be your child's father). Here is an important example of an equivalence relation on $\mathbb{Z}$. For m a positive integer, we will say that

$$a \sim b \text{ if } m \text{ divides } a - b.$$

In this situation we say that a and b are equivalent "mod m". In the exercises you will show that this is indeed an equivalence relation.

Definition 3. Let $\sim$ be an equivalence relation on a set A. If $a \in A$, the *equivalence class* of a, denoted by $[a]$, is the set consisting of those elements of A which are equivalent to a; in set notation this means

$$[a] = \{b \in A \mid a \sim b\}.$$

The set of distinct equivalence classes is denoted by $A/\sim$, pronounced "A mod $\sim$".

Note that each equivalence class is nonempty since $[a]$ in particular contains the element a by the reflexive property of an equivalence relation.

Let us return to the example given above. We will take $m = 2$ and compute the equivalence class of 0 mod 2. Using the definition of an equivalence class, we are looking for all integers k such that $0 \sim k$. By the definition of this equivalence relation, this means are looking for all k so that

$$2 \text{ divides } 0 - k.$$

But this condition just means that $0 - k$ is even, which requires that k also be even. Hence the equivalence class of 0 is just the set of all even integers!

Now, what about the equivalence class of 1? Here we are looking for all integers k such that

$$2 \text{ divides } 1 - k.$$

This condition just means that $1 - k$ is even, which means that k must be odd. Hence the equivalence class of 1 mod 2 is just the set of odd integers. Finally, let us look at the equivalence class of 2. Here we are looking for integers k so that 2 divides $2 - k$. This means that $2 - k$ would have to be even, so k would also have to be even. Thus the equivalence class of 2 is again the set of even integers; i.e. $[0] = [2]$.

Since we now know that 0 and 2 give the same equivalence class, we can ask whether or not there are any more distinct equivalence classes. The answer is given by:

Proposition 1. *Let $\sim$ be an equivalence relation on a set A. Then $[a] = [b]$ if and only if $a \sim b$.*

Proof. First, suppose that $[a] = [b]$. Since $b \in [b]$, b is also in $[a]$. By the definition of the equivalence class of a, this means that $a \sim b$.

Conversely, suppose that $a \sim b$. We want to show that the sets $[a]$ and $[b]$ are equal. By symmetry, it suffices to show that $[a] \subseteq [b]$. To this end, let $c \in [a]$. Then $a \sim c$ by the definition of an equivalence class. By the symmetric property of an equivalence class, $c \sim a$, and by the transitive property we conclude that $c \sim b$. Hence $b \sim c$ so $c \in [b]$. Thus $[a] \subseteq [b]$. $\square$

So, in the example above, since we already know that [1] is the set of odd integers, we know that $[1] = [3] = [5] = [k]$ for any odd integer k. Similarly, the equivalence class of any even integer is just the set of all even integers. Hence this equivalence relation only has two equivalence classes; i.e. $\mathbb{Z}/\sim$ has two elements.

Note that the equivalence classes in this example are disjoint, and that their union is all of $\mathbb{Z}$. This is true in general, and in this sense we can think of an equivalence relation as "breaking a set up into pieces". The following makes this notion precise.

Definition 4. A *partition* of a set A is a collection of subsets of A which are pairwise disjoint and whose union is all of A.

Theorem 1. *The distinct equivalence classes of an equivalence relation on a set A form a partition of A; conversely, any partition of a set A arises as the equivalence classes of some equivalence relation.*

Proof. Left as an exercise. □

So for example, the set of even integers and the set of odd integers forms a partition of $\mathbb{Z}$, which as we saw comes from the equivalence relation given by equivalence mod 2. For any m, the set of equivalence classes mod m of $\mathbb{Z}$ is usually denoted by $\mathbb{Z}_m$ or $\mathbb{Z}/m\mathbb{Z}$. You will see these as basic examples of what are called "groups" when you take Math 113. For fun, try to think about how you might try to define the "sum" of two equivalence classes mod m of $\mathbb{Z}$.

Exercises

1. Give examples of relations which are: (a) reflexive and symmetric but not transitive, (b) reflexive and transitive but not symmetric, and (c) symmetric but not reflexive nor transitive.

2. Show that the "mod m" relation defined above is an equivalence relation for any positive integer m. Can you describe its equivalence classes in general?

3. Prove Theorem 1. For the converse, suppose that a collection $\{A_i\}$ of subsets of A forms a partition of A. The goal is to define an equivalence relation on A (don't forget to show the relation you define is actually an equivalence relation) whose equivalence classes are exactly the same as the sets A_i. This is actually really easy: to satisfy this condition, when should two elements be equivalent to each other?

4. What equivalence relation on a set A gives rise to the partition consisting of all one element subsets of A? What equivalence relation gives rise to the partition consisting of just the set A itself?