NOTES ON SET THEORY

The purpose of these notes is to cover some set theory terminology not included in Solow's book. You should read Appendix A.1 in the book before reading these notes. The symbol ":=" means that the thing on the left is being defined as the thing on the right.

First, the book defines the notion of the complement, denoted by A^c, of a set A in some universal set U. More generally, we can define the complement of a set inside of any other set:

Definition 1. Let A and B be sets. The *complement of A in B*, denoted by $B \setminus A$ or $B - A$, is the set of elements of B which are not in A; in set notation this means

$$B \setminus A := \{x \in B \mid x \notin A\}.$$

For example, if $\mathbb{Z}$ is the set of integers and $2\mathbb{Z}$ is the set of even integers (this is a common notation), then $\mathbb{Z} \setminus 2\mathbb{Z}$ is the set of odd integers.

We would also like a quick way of saying that two sets have nothing in common. This is given by:

Definition 2. We say that two sets A and B are *disjoint* if they have no elements in common; i.e. if $A \cap B = \emptyset$.

More generally, we will say that a (finite) collection of sets $A_1, \ldots, A_n$ is *pairwise disjoint* if the intersection of any two of them is empty. Note that this is not the same as saying that the intersection

$$A_1 \cap \cdots \cap A_n$$

of all sets in the collection is empty; for example, the intersection of the sets

$$\{1, 2\},\ \{2, 3\},\ \text{and } \{1, 3\}$$

is empty but they are not pairwise disjoint. Although we will not go into infinite collections of sets much, the same definition can be made for these also.

Definition 3. The *Cartesian* (or *Cross*) *product* of two sets A and B, denoted by $A \times B$ and pronounced "A cross B", is the set consisting of ordered pairs (a, b) where $a \in A$ and $b \in B$; in set notation this means

$$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

To be precise, we should give a definition for the term "ordered pair". Such a definition should give rise to the property:

$$(a, b) = (c, d) \text{ if and only if } a = c \text{ and } b = d$$

that we would expect of an ordered pair. We will not focus on this too much, but for completeness sake let us define an ordered pair (a, b) as the set

$$\{\{a\}, \{a, b\}\}.$$

In the exercises you will prove that this satisfies the required property. From now on we will just use ordered pairs as we are used to them and ignore the precise definition.

Definition 4. Let A be a set. The *power set* of A, denoted by $\mathcal{P}(A)$, is the set consisting of all subsets of A; in set notation this means

$$\mathcal{P}(A) := \{S \mid S \text{ is a subset of } A\}.$$

For example, let $A = \{1, 2, 3\}$. Then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Note that the empty set is included in this list; indeed, in the exercises you will prove that the empty set is a subset of any set. In particular, this means that the power set of any set is itself never empty since it at least contains the empty set.

In the example above, A had three elements and $\mathcal{P}(A)$ had $8 = 2^3$ elements. In fact, the following is true in general:

Theorem 1. *If F is a finite set with n elements, then $\mathcal{P}(F)$ is a finite set with 2^n elements.*

Proof. Left as an exercise. □

Because of this theorem, another common notation used for $\mathcal{P}(A)$ is 2^A, even when A is not finite. Later on, we will see a proof of this theorem (which is different than the one the exercise below suggests) using a technique known as induction.

Exercises

1. Prove that the definition

$$(a, b) := \{\{a\}, \{a, b\}\}$$

of an ordered pair given above satisfies the property that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

2. Let A and B be finite sets with m and n elements respectively. How many elements does $A \times B$ have?

3. Prove that the empty set is a subset of any set.

4. Prove Theorem 1 without using induction. Hint: For any subset A of a finite set F, we can define a function f_A from F to the set $\{0, 1\}$ by setting

$$f_A(b) = \begin{cases} 1 \text{ if } b \in A \\ 0 \text{ if } b \notin A. \end{cases}$$

How many such functions are there? Why is the number of such functions equal to the number of subsets of F?