Math 320-1: Final Exam
Northwestern University, Fall 2014

Name: ______________________________

1. (15 points) Give an example of each of the following. You do not have to justify your answer.

(a) A monotone subsequence of the sequence $x_n = \cos \frac{n\pi}{2}$.

(b) A function f on $\mathbb{R}$ which is differentiable only at 0.

(c) A nonnegative, nonconstant integrable function f on $[0, 1]$ such that $\int_0^1 f(x)\,dx = 0$.

(d) A differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that f' is not integrable on $[1, 3]$

(e) A differentiable function f on $(-1, 1)$ such that $f'(x) = |x|$ for all $x \in (-1, 1)$.

2. (15 points) Suppose that $f : (a, b) \to \mathbb{R}$ is continuous and bounded with supremum M. Show that for any $\epsilon > 0$ there exists a **rational** $c \in (a, b)$ such that $M - \epsilon < f(c)$.

3. (10 points) Define the sequence (x_n) by

$$x_n = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \cdots + \frac{\sin n}{n^2} \text{ for } n \geq 1.$$

Show that (x_n) is Cauchy. Hint: For any $n \geq 1$, $\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1}$.

4. (15 points) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^4 \cos \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that f is continuously differentiable at 0 but not twice differentiable at 0.

5. (15 points) Define $f : [-5, 5] \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{x^2} & -5 \leq x < 0 \\ 0 & x = 0 \\ -5\cos x^2 & 0 < x \leq 5. \end{cases}$$

Show that f is integrable on $[-5, 5]$.

6. (15 points) Suppose that $f : [-5,5] \to \mathbb{R}$ is the function from the previous problem:

$$f(x) = \begin{cases} e^{x^2} & -5 \le x < 0 \\ 0 & x = 0 \\ -5\cos x^2 & 0 < x \le 5, \end{cases}$$

and define the function $F : [0,2] \to \mathbb{R}$ by

$$F(x) = \int_0^{x^2} f(t)\,dt.$$

Show that $|F(x) - F(y)| \le 20|x - y|$ for all $x, y \in [0,2]$, and hence that F is uniformly continuous.

7. (15 points) Suppose that $f : [1,2] \to \mathbb{R}$ is continuous and that for any $c \in (1,2)$,

$$3\int_1^c e^x f(x)\,dx - \int_c^2 e^x f(x)\,dx = 0.$$

Show that $f(x) = 0$ for all $x \in [1,2]$. (You **cannot** use the similar problem from the practice final without proof.)