Math 320-1: Final Exam Solutions
Northwestern University, Fall 2014

1. Give an example of each of the following. You do not have to justify your answer.

(a) A monotone subsequence of the sequence $x_n = \cos \frac{n\pi}{2}$.

(b) A function f on $\mathbb{R}$ which is differentiable only at 0.

(c) A nonnegative, nonconstant integrable function f on $[0,1]$ such that $\int_0^1 f(x)\,dx = 0$.

(d) A differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that f' is not integrable on $[1,3]$

(e) A differentiable function f on $(-1,1)$ such that $f'(x) = |x|$ for all $x \in (-1,1)$.

Solution. (a) The subsequence (x_{4n}) of terms indexed by a multiple of 4 is constant, so monotone.

(b) The function defined by $f(x) = x^2$ for $x \in \mathbb{Q}$ and $f(x) = -x^2$ for $x \notin \mathbb{Q}$ works. Note that the function defined similarly but using x and $-x$ instead of x^2 and $-x^2$ does not work since this function is not differentiable at 0: $\lim_{x\to 0}(f(x) - f(0))/(x-0)$ does not exist in this case.

(c) The function which is 0 everywhere except at $\frac{1}{2}$ where $f(\frac{1}{2}) = 1$ works, as does my favorite function.

(d) The function defined by $f(x) = (x-2)^2 \sin \frac{1}{(x-2)^2}$ for $x \neq 2$ and $f(2) = 0$ works, which is just a modification of an example from the practice problems.

(e) The function f defined by $f(x) = \int_{-1}^{x} |t|\,dt$ works by the Second Fundamental Theorem of Calculus. □

2. Suppose that $f : (a,b) \to \mathbb{R}$ is continuous and bounded with supremum M. Show that for any $\epsilon > 0$ there exists a **rational** $c \in (a,b)$ such that $M - \epsilon < f(c)$.

Most of you noted that by one possible characterization of supremums, there exists $x \in (a,b)$ such that $M - \epsilon < f(x)$, but now the point is in guaranteeing that you can actually choose x to be rational. If M was actually a maximum, so that $M = f(p)$ for some $p \in (a,b)$, you can use a sequence of rationals converging to p to get what you need. However, since here f is only continuous on an *open* interval, it is not necessarily true that the supremum M is actually a maximum, so we have to do something different. The denseness of $\mathbb{Q}$ in $\mathbb{R}$ is important, as is the fact that f is continuous since the claim is no longer true if we drop that condition.

Proof. Let $\epsilon > 0$. Since nothing smaller than M can be an upper bound of f, there exists $x \in (a,b)$ such that

$$M - \frac{\epsilon}{2} < f(x).$$

Take a sequence (r_n) of rationals in (a,b) converging to x. Since f is continuous, $f(r_n)$ converges to $f(x)$ so there exists N such that

$$|f(r_N) - f(x)| < \frac{\epsilon}{2}.$$

Then

$$f(r_N) > f(x) - \frac{\epsilon}{2} > \left(M - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} = M - \epsilon,$$

so $r_N \in (a,b)$ is the rational number we want. □

3. Define the sequence (x_n) by

$$x_n = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \cdots + \frac{\sin n}{n^2} \text{ for } n \geq 1.$$

Show that (x_n) is Cauchy. Hint: For any $n \geq 1$, $\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1}$.

Proof. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Let $m \geq n \geq N$ and write m as $m = n + k$ for some $k \geq 0$. Then:

$$\begin{aligned}
|x_m - x_n| &= |x_{n+k} - x_n| \\
&= \left| \frac{\sin(n+1)}{(n+1)^2} + \frac{\sin(n+2)}{(n+2)^2} \cdots + \frac{\sin(n+k)}{(n+k)^2} \right| \\
&\leq \left| \frac{\sin(n+1)}{(n+1)^2} \right| + \left| \frac{\sin(n+2)}{(n+2)^2} \right| \cdots + \left| \frac{\sin(n+k)}{(n+k)^2} \right| \\
&\leq \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} \cdots + \frac{1}{(n+k)^2} \\
&\leq \left(\frac{1}{n} - \frac{1}{n+1} \right) + \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \cdots + \left(\frac{1}{n+k-1} - \frac{1}{n+k} \right) \\
&= \frac{1}{n} - \frac{1}{n+k}
\end{aligned}$$

since all other terms cancel out: the $-\frac{1}{n+1}$ in the first set of parentheses cancels with the $\frac{1}{n+1}$ in the second set, and so on. In the second line we use the fact that the expression for x_{n+k} consists of the sum making up x_n plus the additional terms showing up in the second line. Hence we get:

$$|x_m - x_n| \leq \frac{1}{n} - \frac{1}{n+k} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

showing that (x_n) is Cauchy as required. ◻

4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^4 \cos \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that f is continuously differentiable at 0 but not twice differentiable at 0.

Proof. We first have:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} x^3 \cos \frac{1}{x^2} = 0.$$

Thus f is differentiable at 0 and $f'(0) = 0$. Now, f is differentiable at $x \neq 0$ by the product and chain rules since f agrees with the function $x^4 \cos \frac{1}{x^2}$ everywhere close enough to any $x \neq 0$. Using the product and chain rules we get:

$$f'(x) = \begin{cases} 4x^3 \cos \frac{1}{x^2} + 2x \sin \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Since

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(4x^2 \cos \frac{1}{x^2} + 2x \sin \frac{1}{x^2} \right) = 0 = f'(0).$$

f' is continuous at 0, so f is continuously differentiable at 0.

Finally, we have that

$$\lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0} \frac{f'(x)}{x} = \lim_{x \to 0} \left(4x \cos \frac{1}{x^2} + 2 \sin \frac{1}{x^2} \right)$$

does not exist due to the $\sin \frac{1}{x^2}$ term, so f' is not differentiable at 0, meaning that f is not twice differentiable at 0. ◻

5. Define $f : [-5, 5] \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{x^2} & -5 \le x < 0 \\ 0 & x = 0 \\ -5\cos x^2 & 0 < x \le 5. \end{cases}$$

Show that f is integrable on $[-5, 5]$.

Proof. Set $M = e^{25} + 5$ and note that $|f(x)| \le M$ for all $x \in [-5, 5]$. Let $\epsilon > 0$. On the interval $[-5, \frac{\epsilon}{12M}]$, f agrees with the function e^{x^2}, so it is continuous and hence integrable on this interval. Thus there exists a partition P_1 of this interval such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{3}.$$

On the interval $[\frac{\epsilon}{12M}, 5]$, f agrees with the function $-5\cos x^2$, so it is continuous and hence integrable on this interval as well. Thus there exists a partition P_2 of this interval such that

$$U(f, P_2) - L(f, P_2) < \frac{\epsilon}{3}.$$

Let P be the partition of $[-5, 5]$ consisting of P_1 and P_2. The subintervals determined by P include all the ones determined by P_1 and P_2 as well as the interval $[-\frac{\epsilon}{12M}, \frac{\epsilon}{12M}]$. The contributions from the subintervals making up P_1 to $U(f, P) - L(f, P)$ are less than $\frac{\epsilon}{3}$, as are the contributions from the subinterval making up P_2. On $[-\frac{\epsilon}{12M}, \frac{\epsilon}{12M}]$ we have

$$(\sup f - \inf f)(\text{length}) \le 2M\frac{\epsilon}{6M} = \frac{\epsilon}{3}$$

since $|f(x) - f(y)| \le |f(x)| + |f(y)| \le 2M$ for all $x, y \in [-5, 5]$. Thus we get:

$$U(f, P) - L(f, P) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

showing that f is integrable on $[-5, 5]$ as claimed. $\square$

6. Suppose that $f : [-5, 5] \to \mathbb{R}$ is the function from the previous problem:

$$f(x) = \begin{cases} e^{x^2} & -5 \le x < 0 \\ 0 & x = 0 \\ -5\cos x^2 & 0 < x \le 5, \end{cases}$$

and define the function $F : [0, 2] \to \mathbb{R}$ by

$$F(x) = \int_0^{x^2} f(t)\, dt.$$

Show that $|F(x) - F(y)| \le 20|x - y|$ for all $x, y \in [0, 2]$, and hence that F is uniformly continuous.

Many of you used the Mean Value Theorem here, which works as long as you justify why the Mean Value Theorem is actually applicable! In particular, the integrand f is not continuous on all of $[0, x^2]$, so why is F differentiable? Why is F even continuous on $[0, 2]$? This wasn't a major issue, but was something you had to say something about.

Note that if F was defined using $\int_{-2}^{x^2} f(t)\,dt$ instead, then the Mean Value Theorem is definitely not applicable since now the point at which f is not continuous is in the middle of the region of integration. In this case you have to use an argument like the one I give below. This was actually how I originally envisioned the problem, but some final fiddling with the integral which ended up being used made it so that the Mean Value Theorem was applicable.

Proof. For any $x, y \in [0, 2]$ with $x \geq y$, we have

$$F(x) - F(y) = \int_0^{x^2} f(t)\,dt - \int_0^{y^2} f(t)\,dt = \int_{y^2}^{x^2} f(t)\,dt.$$

Since $0 \leq y^2 \leq x^2$, the function f on the interval $[y^2, x^2]$ is given by $f(t) = -5\cos t^2$, so we get:

$$|F(x) - F(y)| = \left| \int_{y^2}^{x^2} -5\cos t^2\,dt \right| \leq \int_{y^2}^{x^2} \left|-5\cos t^2\right|\,dt \leq \int_{y^2}^{x^2} 5\,dt = 5|x^2 - y^2|.$$

Since $0 \leq x, y \leq 2$, $|x + y| \leq 4$, so

$$|F(x) - F(y)| \leq 5|x^2 - y^2| = 5|x + y||x - y| \leq 20|x - y|$$

as claimed. To see that F is uniformly continuous, for $\epsilon > 0$ let $\delta = \frac{\epsilon}{20}$. Then if $|x - y| < \delta$ with $x, y \in [0, 2]$, we have

$$|F(x) - F(y)| \leq 20|x - y| < 2\delta = \epsilon$$

as required. $\square$

7. Suppose that $f : [1, 2] \to \mathbb{R}$ is continuous and that for any $c \in (1, 2)$,

$$3\int_1^c e^x f(x)\,dx - \int_c^2 e^x f(x)\,dx = 0.$$

Show that $f(x) = 0$ for all $x \in [1, 2]$.

This is a simplified version of the last problem on the practice final. One thing to note is that taking derivatives with respect to c will in the end only show that $f(x) = 0$ for $x \in (1, 2)$, and you then have to use continuity of f to get that $f(1) = 0 = f(2)$ as well.

Proof. For $c \in (1, 2)$, the function

$$F(c) = 3\int_1^c e^x f(x)\,dx + \int_2^c e^x f(x)\,dx$$

is differentiable with respect to c by the Second Fundamental Theorem of Calculus. The given condition says that $F(c) = 0$ for all $c \in (1, 2)$, so differentiating with respect to c gives:

$$0 = F'(c) = 3e^c f(c) + e^c f(c) = 4e^c f(c).$$

Since $4e^c \neq 0$, we must have $f(c) = 0$ for all $c \in (1, 2)$. Since f is continuous,

$$f(1) = \lim_{x \to 1} f(x) = 0 \text{ and } f(2) = \lim_{x \to 1} f(x) = 0$$

as well, so $f(x) = 0$ for all $x \in [1, 2]$. $\square$